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Interfaces for Random Cluster Models

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A random cluster measure on $\mathbb{Z}^d$ that is not translationally invariant is constructed for $d \geqslant 3$, the critical density p_c, and sufficiently large q. The resulting measure is proven to be a Gibbs state satisfying cluster model DLR- equations.

KEY WORDS: Random cluster; interface; graphical representation; Pirogov–Sinai.

1. INTRODUCTION

In this paper we study the existence of Gibbs measures for a random cluster model on $\mathbb{Z}^d$, $d \geqslant 3$, that are not translation-invariant. In particular, they are not a linear combination of μ^o and μ^v, the measures constructed as limits with fully occupied (wired) and vacant (empty) boundary conditions, respectively. A translation noninvariant measure is obtained, for the random cluster model at the critical occupation density p_c and with sufficiently large q, as a limit of the measures in finite boxes with Dobrushin type[(1)] boundary conditions: vacant in a halfspace and occupied in the complementary one. In the language of the corresponding Potts model, with an integer q, this corresponds to the coexistence of the ordered and disordered phases. After having constructed the limiting measure, we show some of its properties and prove that it is indeed a Gibbs state.

While the existence of such translation noninvariant states has been more or less explicitly stated (especially in the context of the Potts model) and the idea of the limiting construction was sketched,[(2–4)] the details of the proof have never been presented. Having also in mind that it was

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conjectured that for a general random cluster model a translation noninvariant Gibbs measure does not exist[5] and that the subtle problem of proving that the limiting measure is, indeed, a Gibbs state in the sense of satisfying the DLR equations for random cluster measures (the relevant definitions will be recalled below), we considered it useful to present a detailed proof of these facts.

2. SETTINGS AND RESULTS

We begin with some notation. For any subset $\Lambda \subset \mathbb{Z}^d$ we introduce $\mathbb{B}_0(\Lambda)$ as the set of all bonds $b = \langle x, y \rangle$ of nearest neighbours with both endpoints in Λ, the set $\mathbb{B}(\Lambda)$ as the set of all bonds with at least one endpoint in Λ and $\partial\mathbb{B}(\Lambda)$ as $\mathbb{B}(\Lambda) \setminus \mathbb{B}_0(\Lambda)$. For any $\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)$ we define the set $\mathbb{V}(\mathbb{B})$ as the set of sites which belong to at least one bond in $\mathbb{B}$. The relevant sample space for random cluster model is the set $\Omega = \{0, 1\}^{\mathbb{B}(\mathbb{Z}^d)}$ of configurations $\boldsymbol{\eta} = \{\eta_b\}$, $b \in \mathbb{B}(\mathbb{Z}^d)$, with $\eta_b \in \{0, 1\}$. For any configuration $\boldsymbol{\eta}$ we introduce the set of all *occupied* bonds $\mathbb{B}(\boldsymbol{\eta}) = \{b \in \mathbb{B}(\mathbb{Z}^d) : \eta(b) = 1\}$ and the corresponding graph $(\mathbb{Z}^d, \mathbb{B}(\boldsymbol{\eta}))$ with the vertex set $\mathbb{Z}^d$ and the edge set $\mathbb{B}(\boldsymbol{\eta})$. We write $\boldsymbol{\eta}_{\mathbb{B}}$ for $\{\eta_b\}_{b \in \mathbb{B}}$ and $\boldsymbol{\eta}_{\mathbb{B}} \circ \bar{\boldsymbol{\eta}}_{\mathbb{B}^c}$ for the configuration that equals to $\boldsymbol{\eta}_{\mathbb{B}}$ on $\mathbb{B}$ and to $\bar{\boldsymbol{\eta}}_{\mathbb{B}^c}$ on $\mathbb{B}^c$.

We define the *random cluster measure* on a finite set $\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)$ as follows. Let $0 \leqslant p \leqslant 1$, $q > 0$, and let $\bar{\boldsymbol{\eta}}_{\mathbb{B}^c}$ be a fixed configuration on $\mathbb{B}^c$. For any configuration $\boldsymbol{\eta}$, let $k_{\mathbb{B}}(\boldsymbol{\eta})$ be the number of components $C(\boldsymbol{\eta})$ (including isolated vertices) of the graph $(\mathbb{Z}^d, \mathbb{B}(\boldsymbol{\eta}))$ such that the vertex set $\mathbb{V}(C(\boldsymbol{\eta}))$ intersects $\mathbb{V}(\mathbb{B})$. The random cluster measure on $\mathbb{B}$ with boundary condition $\bar{\boldsymbol{\eta}}_{\mathbb{B}^c}$, and with fixed parameters p and q, is given by

$$\mu_{\mathbb{B}}^{\bar{\boldsymbol{\eta}}}(\boldsymbol{\eta}) = \frac{1}{Z_{\mathbb{B}}^{\bar{\boldsymbol{\eta}}}} p^{|\mathbb{B}(\boldsymbol{\eta}) \cap \mathbb{B}|}(1-p)^{|\mathbb{B} \setminus \mathbb{B}(\boldsymbol{\eta})|} q^{k_{\mathbb{B}}(\boldsymbol{\eta}_{\mathbb{B}} \circ \bar{\boldsymbol{\eta}}_{\mathbb{B}^c})} \tag{2.1}$$

if $\boldsymbol{\eta} = \bar{\boldsymbol{\eta}}$ on $\mathbb{B}^c$ and $\mu_{\mathbb{B}}^{\bar{\boldsymbol{\eta}}}(\boldsymbol{\eta}) = 0$ otherwise. Here, $Z_{\mathbb{B}}(\bar{\boldsymbol{\eta}})$ is the *partition function*,

$$Z_{\mathbb{B}}^{\bar{\boldsymbol{\eta}}} = \sum_{\boldsymbol{\eta}_{\mathbb{B}}} p^{|\mathbb{B}(\boldsymbol{\eta}) \cap \mathbb{B}|}(1-p)^{|\mathbb{B} \setminus \mathbb{B}(\boldsymbol{\eta})|} q^{k_{\mathbb{B}}(\boldsymbol{\eta}_{\mathbb{B}} \circ \bar{\boldsymbol{\eta}}_{\mathbb{B}^c})}, \tag{2.2}$$

with the sum running over all configurations $\boldsymbol{\eta}$ equal to $\bar{\boldsymbol{\eta}}$ on $\mathbb{B}^c$. To avoid heavy notation, we do not explicitly mark the dependence on the parameters p and q.

As usual, there are two natural candidates for infinite volume random cluster measure. One can either use Dobrushin–Lanford–Ruelle (DLR) equations or take a weak limit over finite boxes.

Definition 2.1. A probability measure μ on $\{0, 1\}^{\mathbb{B}(\mathbb{Z}^d)}$ is called the Gibbs random cluster measure if

$$\mu(f) = \int \mu(d\boldsymbol{\eta})\, \mu_{\mathbb{B}}^{\boldsymbol{\eta}}(f) \tag{2.3}$$

for any finite $\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)$ and any cylinder function f with support in $\mathbb{B}$.

Definition 2.2. A probability measure μ on $\{0, 1\}^{\mathbb{B}(\mathbb{Z}^d)}$ is called a limit random cluster measure if there exists an increasing sequence $\mathbb{B}_1 \subset \mathbb{B}_2 \subset \cdots$ of bond sets such that $\bigcup_{n=1}^{\infty} \mathbb{B}_n = \mathbb{B}(\mathbb{Z}^d)$ and a configuration $\boldsymbol{\eta} \in \Omega$ such that

$$\lim_{n \to \infty} \mu_{\mathbb{B}_n}^{\boldsymbol{\eta}}(f) = \mu(f) \tag{2.4}$$

for any cylinder function f.

Let us now sum up some known facts about random cluster model. For more detailed discussion see, for example, refs. 4, 6, and 7. Let $\boldsymbol{\eta}^o$ ($\boldsymbol{\eta}^v$) be the fully occupied (vacant) configuration defined by $\eta_b^o = 1$ ($\eta_b^v = 0$) for all $b \in \mathbb{B}(\mathbb{Z}^d)$. It is known that the measures

$$\mu^o(\cdot) = \lim_{\mathbb{B} \to \mathbb{B}(\mathbb{Z}^d)} \mu_{\mathbb{B}}^{\boldsymbol{\eta}^o}(\cdot) \qquad \text{and} \tag{2.5}$$

$$\mu^v(\cdot) = \lim_{\mathbb{B} \to \mathbb{B}(\mathbb{Z}^d)} \mu_{\mathbb{B}}^{\boldsymbol{\eta}^v}(\cdot) \tag{2.6}$$

exist and are Gibbs random cluster measures. In addition, for any $p \in [0, 1]$, $q \geqslant 1$, the inequality $\mu^v \leqslant_{\mathrm{FKG}} \mu \leqslant_{\mathrm{FKG}} \mu^o$ holds for any Gibbs random cluster measure μ. Further, for q large enough, there exists $p_c \in (0, 1)$ such that the measures μ^o and μ^v differ for $p = p_c$. Moreover, for all $p \neq p_c$ the measures μ^o and μ^v coincide and thus there is only one Gibbs measure.

To state our main theorem we need the following notation. Let $\boldsymbol{\xi} \in \Omega$ be the configuration defined as follows: $\xi_b = 1$ for all bonds b whose both end-vertices have a nonnegative dth coordinate and $\xi_b = 0$ otherwise. Further, let $\Lambda_{L,M}$ be the box $([-L, L]^{d-1} \times [-M, M]) \cap \mathbb{Z}^d$ and $\mathbb{B}_{L,M}$ be the set $\mathbb{B}_0(\Lambda_{L,M}) \cup (\partial\mathbb{B}(\Lambda_{L,M}) \cap \mathbb{B}(\boldsymbol{\xi}))$. We will write Λ_L for $\Lambda_{L,\infty}$ and $\mathbb{B}_L$ for $\mathbb{B}_{L,\infty}$.

Theorem 2.3. Let $d \geqslant 3$. There exists $q_0 = q_0(d)$ such that if $q \geqslant q_0$ and $p = p_c$, then:

(i) The limit measure

$$\mu^{\xi}(\cdot) = \lim_{L\to\infty} \lim_{M\to\infty} \mu^{\xi}_{\mathbb{B}_{L,M}}(\cdot) \tag{2.7}$$

exists.

(ii) There is, μ^{ξ}-a.s., exactly one infinite cluster.

(iii) The measure μ^{ξ} is not translation invariant.

(iv) The measure μ^{ξ} is a Gibbs random cluster measure.

The outline of the proof follows. We first rewrite the random cluster partition function in a finite box using a contour representation. Secondly, we use this representation to describe the the extremal Gibbs measures μ^{o} and μ^{v}. Next, we apply the methods from[8] (that are in itself an extension of the Dobrushin proof of existence of a translation noninvariant Gibbs state for the Ising model) to construct the limit measure μ^{ξ}. We will see that there is a rigid interface separating an infinite occupied cluster in the upper half space from the region with only finite clusters in the lower half-space. Then, we use the uniqueness of the infinite cluster to prove that limit measure μ^{ξ} is a Gibbs random cluster measure.

3. INTRODUCTION OF CONTOURS

For an arbitrary configuration $\boldsymbol{\eta} \in \Omega$, we first consider the set $B(\boldsymbol{\eta}) \subset \mathbb{R}^d$ obtained as the union of all bonds from $\mathbb{B}(\boldsymbol{\eta})$ viewed as unit segments in $\mathbb{R}^d$. Further, we extend it and define the set $F(\boldsymbol{\eta})$ as the union of $B(\boldsymbol{\eta})$ with all unit squares whose all four edges are occupied as well as with all unit cubes whose all twelve edges are in $B(\boldsymbol{\eta})$, etc. Now, taking the 1/4-neighbourhood $U_{1/4}(F(\boldsymbol{\eta}))$ of $F(\boldsymbol{\eta})$ (in maximal norm), we define the *contours* as finite components of the boundary of $U_{1/4}(F(\boldsymbol{\eta}))$.

Let us note, that for any contour γ there exists one infinite component of $\mathbb{R}^d \setminus \gamma$ denoted $\operatorname{Ext}\gamma$ and let $\operatorname{Int}\gamma = \mathbb{R}^d \setminus (\operatorname{Ext}\gamma \cup \gamma)$. For any contour γ there exists a unique configuration $\boldsymbol{\eta}$, such that γ is the only contour of $\boldsymbol{\eta}$. The contour γ is called *o-contour*, if all bonds in $\mathbb{B}_0(\operatorname{Ext}\gamma \cap \mathbb{Z}^d)$ are occupied in this configuration, otherwise γ is called *v-contour*. We use $\mathscr{K}_o$ (resp. $\mathscr{K}_v$) to denote the set of all o-contours (resp. v-contours), and set $\mathscr{K} = \mathscr{K}_o \cup \mathscr{K}_v$. A collection $\partial \subset \mathscr{K}$ of contours is *compatible*, if there exists $\boldsymbol{\eta} \in \Omega$, such that ∂ matches with the boundary of $U_{1/4}(F(\boldsymbol{\eta}))$. We use the symbol $\boldsymbol{\eta}(\partial)$ to denote this configuration. Let $\mathscr{D}$ denote the set of all compatible collections of contours. We say that a contour $\gamma \in \partial$ is an *external* contour of a compatible collection ∂ if $\gamma \not\subset \operatorname{Int}\bar{\gamma}$ for all $\bar{\gamma} \in \partial$. We use $\mathscr{D}^e$ to denote the set of all compatible collections with all contours external, $\mathscr{D}_{\mathbb{B},\alpha}$,

$\alpha = o, v$, to denote the set of all compatible families ∂ of contours such that the corresponding configuration $\boldsymbol{\eta}(\partial)$ coincides with $\boldsymbol{\eta}^{\alpha}$ outside the set $\mathbb{B}$, and we set $\mathscr{D}_{\alpha} = \bigcup_{\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)} \mathscr{D}_{\mathbb{B}, \alpha}$. Further, we use $\|\gamma\|$ to denote the number of intersections of γ with bonds from $\mathbb{B}(\mathbb{Z}^d)$ and $\rho(\gamma)$ to denote the *weight* of the contour,

$$\rho(\gamma) = \begin{cases} q^{-\|\gamma\|/2d} & \text{if} \quad \gamma \text{ is } o\text{-contour}, \\ qq^{-\|\gamma\|/2d} & \text{if} \quad \gamma \text{ is } v\text{-contour}. \end{cases} \tag{3.1}$$

Finally, we introduce the *energies* of occupied and vacant bonds $e_o = -\log p$ and $e_v = -\log(1-p) - (\log q)/d$.

Lemma 3.1. For any box Λ we have:

(a) $$Z^{\boldsymbol{\eta}^v}_{\mathbb{B}_0(\Lambda)} = q^{\frac{|\partial \mathbb{B}(\Lambda)|}{2d}} \sum_{\partial \in \mathscr{D}_{\mathbb{B}_0(\Lambda), v}} e^{-e_o\, |\mathbb{B}(\boldsymbol{\eta}(\partial))|} e^{-e_v\, |\mathbb{B}_0(\Lambda) \setminus \mathbb{B}(\boldsymbol{\eta}(\partial))|} \prod_{\gamma \in \partial} \rho(\gamma), \tag{3.2}$$

(b) $$Z^{\boldsymbol{\eta}^o}_{\mathbb{B}(\Lambda)} = q \sum_{\partial \in \mathscr{D}_{\mathbb{B}(\Lambda), o}} e^{-e_o\, |\mathbb{B}(\boldsymbol{\eta}(\partial)) \cap \mathbb{B}(\Lambda)|} e^{-e_v\, |\mathbb{B}(\Lambda) \setminus \mathbb{B}(\boldsymbol{\eta}(\partial))|} \prod_{\gamma \in \partial} \rho(\gamma). \tag{3.3}$$

Proof. We say that a site $x \in \mathbb{Z}^d$ is $\boldsymbol{\eta}$-isolated if all bonds incident to x are vacant. Let $E_\Lambda(\boldsymbol{\eta})$ be the number of $\boldsymbol{\eta}$-isolated bonds in Λ. For any $\boldsymbol{\eta}$ we define the sets

$$\partial_i \boldsymbol{\eta} = \{b \in \mathbb{B}(\mathbb{Z}^d) \setminus \mathbb{B}(\boldsymbol{\eta}) : |\mathbb{V}(b) \cap \mathbb{V}(\mathbb{B}(\boldsymbol{\eta}))| = i\}, \quad i = 1, 2. \tag{3.4}$$

The following equalities are obvious,

$$E_\Lambda(\boldsymbol{\eta}) = |\Lambda| - |\Lambda \cap \mathbb{V}(\mathbb{B}(\boldsymbol{\eta}))|, \tag{3.5}$$

$$2d\, |\Lambda| = 2\, |\mathbb{B}_0(\Lambda)| + |\partial \mathbb{B}(\Lambda)|. \tag{3.6}$$

Now, to prove (a) let $\boldsymbol{\eta}$ be any configuration identical to $\boldsymbol{\eta}^v$ outside $\mathbb{B}_0(\Lambda)$. Then we have:

$$2d\, |\Lambda \cap \mathbb{V}(\mathbb{B}(\boldsymbol{\eta}))| = 2\, |B(\boldsymbol{\eta})| + 2\, |\partial_2 \boldsymbol{\eta}| + |\partial_1 \boldsymbol{\eta}|. \tag{3.7}$$

In terms of the collection $\partial(\boldsymbol{\eta})$ of contours of $\boldsymbol{\eta}$, we have

$$2\, |\partial_2 \boldsymbol{\eta}| + |\partial_1 \boldsymbol{\eta}| = \sum_{\gamma \in \partial(\boldsymbol{\eta})} \|\gamma\| \tag{3.8}$$

and

$$k_{\mathbb{B}_0(\Lambda)}(\boldsymbol{\eta}) = |\partial(\boldsymbol{\eta}) \cap \mathscr{K}_v| + E_\Lambda(\boldsymbol{\eta}), \tag{3.9}$$

with $|\partial(\boldsymbol{\eta}) \cap \mathscr{K}_v|$ denoting the number of v-contours in $\partial(\boldsymbol{\eta})$. Combining (3.5)–(3.9) with definition of partition function we get (a).

To prove (b), we proceed similarly. Let $\boldsymbol{\eta}$ be any configuration identical to $\boldsymbol{\eta}^o$ outside $\mathbb{B}(\Lambda)$. Then

$$\begin{aligned}2d\,|\Lambda \cap \mathbb{V}(\mathbb{B}(\boldsymbol{\eta}))| = 2\,&|\mathbb{B}(\boldsymbol{\eta}) \cap \mathbb{B}_0(\Lambda)| + |\mathbb{B}(\boldsymbol{\eta}) \cap \partial\mathbb{B}(\Lambda)| \\ &+ 2\,|\partial_2\boldsymbol{\eta} \cap \mathbb{B}_0(\Lambda)| + |\partial_2\boldsymbol{\eta} \cap \partial\mathbb{B}(\Lambda)| + |\partial_1\boldsymbol{\eta} \cap \mathbb{B}_0(\Lambda)|.\end{aligned} \tag{3.10}$$

In terms of the contour representation,

$$\begin{aligned}\sum_{\gamma \in \partial(\boldsymbol{\eta})} &\|\gamma\| - |\partial\mathbb{B}(\Lambda) \setminus \mathbb{B}(\boldsymbol{\eta})| \\ &= 2\,|\partial_2\boldsymbol{\eta} \cap \mathbb{B}_0(\Lambda)| + |\partial_2\boldsymbol{\eta} \cap \partial\mathbb{B}(\Lambda)| + |\partial_1\boldsymbol{\eta} \cap \mathbb{B}_0(\Lambda)|\end{aligned} \tag{3.11}$$

and

$$k_{\mathbb{B}(\Lambda)}(\boldsymbol{\eta}) = 1 + |\partial(\boldsymbol{\eta}) \cap \mathscr{K}_v| + E_\Lambda(\boldsymbol{\eta}). \tag{3.12}$$

After a similar calculation as for (a) we get (b). ∎

For any contour γ we define $\mathbb{B}_{\mathrm{Int}\,\gamma}$ as the set of all bonds having their centres in $\mathrm{Int}\,\gamma$ and introduce the "partition sums" $Z^v(\mathrm{Int}\,\gamma)$ and $Z^o(\mathrm{Int}\,\gamma)$:

$$Z^v(\mathrm{Int}\,\gamma) = \sum_{\substack{\partial \in \mathscr{D}_v \\ \partial \subset \mathrm{Int}\,\gamma}} e^{-e_o\,|\mathbb{B}(\boldsymbol{\eta}(\partial)) \cap \mathbb{B}_{\mathrm{Int}\,\gamma}|} e^{-e_v\,|\mathbb{B}_{\mathrm{Int}\,\gamma} \setminus \mathbb{B}(\boldsymbol{\eta}(\partial))|} \prod_{\bar{\gamma} \in \partial} \rho(\bar{\gamma}), \tag{3.13}$$

$$Z^o(\mathrm{Int}\,\gamma) = \sum_{\substack{\partial \in \mathscr{D}_o \\ \partial \subset \mathrm{Int}\,\gamma}} e^{-e_o\,|\mathbb{B}(\boldsymbol{\eta}(\partial)) \cap \mathbb{B}_{\mathrm{Int}\,\gamma}|} e^{-e_v\,|\mathbb{B}_{\mathrm{Int}\,\gamma} \setminus \mathbb{B}(\boldsymbol{\eta}(\partial))|} \prod_{\bar{\gamma} \in \partial} \rho(\bar{\gamma}). \tag{3.14}$$

Here $\partial \subset \mathrm{Int}\,\gamma$ is a shorthand for $\bar{\gamma} \subset \mathrm{Int}\,\gamma$ for each $\bar{\gamma} \in \partial$. Notice that since $\mathrm{Int}\,\gamma \subset \mathbb{R}^d$ is open, a contour $\bar{\gamma} \subset \mathrm{Int}\,\gamma$ does not intersect the boundary of $\mathrm{Int}\,\gamma$. Further, we use this notation to define contour functionals $\Phi_o\colon \mathscr{K}_o \to \mathbb{R}$ and $\Phi_v\colon \mathscr{K}_v \to \mathbb{R}$ by

$$\Phi_v(\gamma) = \rho(\gamma)\,\frac{Z^v(\mathrm{Int}\,\gamma)}{Z^o(\mathrm{Int}\,\gamma)}, \tag{3.15}$$

$$\Phi_o(\gamma) = \rho(\gamma)\,\frac{Z^o(\mathrm{Int}\,\gamma)}{Z^v(\mathrm{Int}\,\gamma)}. \tag{3.16}$$

Their importance lies in the fact that they allow to use the powerful theory of polymer models.

This is done in two steps. First, it can be shown that, for $p = p_c$ and q sufficiently large, these functionals satisfy the bounds

$$|\Phi_\alpha(\gamma)| \leqslant e^{-\tau \|\gamma\|} \tag{3.17}$$

for each $\gamma \in \mathscr{K}_\alpha$, $\alpha = v, o$, with $\tau = \frac{1}{2d} \log q - \tau_0$ with a fixed $\tau_0 = \tau_0(d)$. This step needs an employment of the full Pirogov–Sinai theory[9] and we refer to the expositions[10] where this result is presented exactly in the current setting. Actually, the main claim of the Pirogov–Sinai theory in the present context is that there exists the value p_c such that (3.17) is valid for Φ_v whenever $p \leqslant p_c$ and for Φ_o whenever $p \geqslant p_c$. This value p_c turns out[7] to be identical to the percolation threshold.

Once the bound (3.17) is established, the functionals Φ_o and Φ_v can be used to control the partition functions (3.2) and (3.3) in terms of polymer models with convergent cluster expansions.[3] The main claims of the theory of polymer models are, for the reader convenience, summarised in Appendix B. In particular, whenever $\mathscr{L} \subset \mathscr{K}_\alpha$ is a finite set of contours, we define

$$\mathscr{Z}(\mathscr{L}, \Phi_\alpha) = \sum_{\partial \subset \mathscr{L}} \prod_{\gamma \in \partial} \Phi_\alpha(\gamma), \tag{3.18}$$

where the sum is over sets ∂ of mutually compatible contours. As stated in Theorem 2.2, assuming that Φ_α satisfies (3.17), there exists a map Φ^T_α assigning a complex number $\Phi^T_\alpha(C)$ to each finite $C \subset \mathscr{K}_\alpha$, $\Phi^T_\alpha : \exp(\mathscr{K}_\alpha) \mapsto \mathbb{C}$, such that

$$\log Z(\mathscr{L}, \Phi_\alpha) = \sum_{C \subset \mathscr{L}} \Phi^T_\alpha(C) \tag{3.19}$$

for any finite $\mathscr{L} \subset \mathscr{K}_\alpha$. In addition, there exists $\omega_0 = \omega_0(d)$ such that for for every $x \in \mathbb{R}^d$, the estimate

$$\sum_{\substack{C \subset \mathscr{K}_\alpha \\ \bigcup_{\gamma \in C} \gamma \ni x}} |\Phi^T_\alpha(C)| \, e^{\omega \|C\|} \leqslant 1 \tag{3.20}$$

[3] The term cluster is used in two different meanings. First, we deal with random *cluster* models, while here we employ *cluster* expansions for their control. Since both notions are well established and used in the literature, we will use them simultaneously—which one we have in mind will always be clear from the context.

is satisfied with $\omega=\frac{1}{2d}\log q-\omega_0$. We use $\|C\|$ to denote the overall length of the cluster C, $\|C\|=\sum_{\gamma\in C}\|\gamma\|$.

Using now $\mathscr{K}_v(\Lambda)$ and $\mathscr{K}_o(\Lambda)$ to denote the sets of v-contour and o-contours within 1-neighbourhood of Λ, we rewrite the partition functions (3.2) and (3.3) in terms of partition functions (3.18) of corresponding polymer models.

Lemma 3.2. Let Λ be an arbitrary box in $\mathbb{Z}^d$. Then

(a) $$Z^{\mathbf{\eta}^v}_{\mathbb{B}_0(\Lambda)}=q^{|\partial\mathbb{B}(\Lambda)|/2d}\exp(-e_v\,|\mathbb{B}_0(\Lambda)|)\,\mathscr{Z}(\mathscr{K}_v(\Lambda),\Phi_v),\tag{3.21}$$

(b) $$Z^{\mathbf{\eta}^o}_{\mathbb{B}(\Lambda)}=q\exp(-e_o\,|\mathbb{B}(\Lambda)|)\,\mathscr{Z}(\mathscr{K}_o(\Lambda),\Phi_o).\tag{3.22}$$

Proof. We prove here (a), the claim (b) can be proven in similar way. For any $\partial\in\mathscr{D}_{\mathbb{B}_0(\Lambda),v}$ we use $\Theta(\partial)$ to denote the set of external contours of ∂ and $\operatorname{Ext}\partial$ to denote $\bigcap_{\gamma\in\Theta(\partial)}\operatorname{Ext}\gamma$. Let us write $\mathbb{B}_{\operatorname{Ext}\partial}(\Lambda)$ for $\mathbb{B}_0(\Lambda)\setminus(\bigcup_{\gamma\in\partial}\mathbb{B}_{\operatorname{Int}\gamma})$. For any $\gamma\in\Theta(\partial)$ we define the set $\bar{\partial}(\gamma)=\{\bar{\gamma}\in\partial:\bar{\gamma}\subset\operatorname{Int}\gamma\}$.

Since all external contours of ∂ are v-contours we can rewrite (3.2) in the following way:

$$\begin{aligned}
Z^{\mathbf{\eta}^v}_{\mathbb{B}_0(\Lambda)}&=q^{|\partial\mathbb{B}(\Lambda)|/2d}\sum_{\partial\in\mathscr{D}_{\mathbb{B}_0(\Lambda),v}}\exp[-e_v\,|\mathbb{B}_{\operatorname{Ext}\partial}(\Lambda)|]\\
&\quad\times\prod_{\gamma\in\Theta(\partial)}\exp[-e_o\,|\mathbb{B}_{\operatorname{Int}\gamma}\cap\mathbb{B}(\mathbf{\eta}(\bar{\partial}(\gamma)))|-e_v\,|\mathbb{B}_{\operatorname{Int}\gamma}\setminus\mathbb{B}(\mathbf{\eta}(\bar{\partial}(\gamma)))|]\prod_{\gamma\in\partial}\rho(\gamma)\\
&=q^{|\partial\mathbb{B}(\Lambda)|/2d}\sum_{\Theta}\exp[-e_v\,|\mathbb{B}_{\operatorname{Ext}\partial}(\Lambda)|]\prod_{\gamma\in\Theta}\rho(\gamma)\,Z^v(\operatorname{Int}\gamma)\,\frac{Z^o(\operatorname{Int}\gamma)}{Z^v(\operatorname{Int}\gamma)},
\end{aligned}\tag{3.23}$$

where the sum runs over all collections Θ of mutually external contours from $\mathscr{K}_v(\Lambda)$.

We can use (3.16) and iterate this step by expanding $Z^v(\operatorname{Int}\gamma)$. The number of iterations will be necessarily finite, since Λ is a finite box. After all iterations we get (a). ∎

4. UNIQUENESS OF THE LIMIT RANDOM CLUSTER MEASURE IN A FINITE BASE CYLINDER

The proposition proved here is used only in the proof of the existence of limit measure μ^{ξ}. However, it can be of its own interest. Let Λ be any finite base cylinder in $\mathbb{Z}^d$, i.e., the $\Lambda=Q\times\mathbb{Z}$ for some finite set $Q\subset\mathbb{Z}^{d-1}$

and let $\mathbb{B}$ be such that $\mathbb{B}_0(\Lambda) \subset \mathbb{B} \subset \mathbb{B}(\Lambda)$. We use the symbol $\mathbb{B}_m^n$ for the subset of $\mathbb{B}$ within the slab $\mathbb{R}^{d-1} \times [m, n]$.

The configuration $\boldsymbol{\eta}$ is called $\mathbb{B}$-good boundary condition if for every noninteger $h \in \mathbb{R}$ there are no two different components of $\mathbb{B}(\boldsymbol{\eta}) \cap \mathbb{B}^c$ viewed as a graph, such that both contain vertices from $\mathbb{V}(\mathbb{B})$ above as well as bellow the height h.

Proposition 4.1. Let $\boldsymbol{\eta}$ be a $\mathbb{B}$-good boundary condition, $0 < p < 1$ and $q \geqslant 1$. Then the limit random cluster measure $\mu_{\mathbb{B}}^{\boldsymbol{\eta}} = \lim_{n \to \infty} \mu_{\mathbb{B}_{-n}^n}^{\boldsymbol{\eta}}$ exists and is independent of the values of $\boldsymbol{\eta}$ on $\mathbb{B}$.

Proof. Let us write $\boldsymbol{\eta}^{(\alpha)}$ as the configuration $\boldsymbol{\eta}_{\mathbb{B}}^{\alpha} \circ \boldsymbol{\eta}_{\mathbb{B}^c}$, $\alpha = o, v$. It is not difficult to prove using the FKG inequality[6, 7] that $\mu_{\mathbb{B}'}^{\boldsymbol{\eta}^{(v)}} \leqslant_{\mathrm{FKG}} \mu_{\mathbb{B}''}^{\boldsymbol{\eta}^{(v)}}$, whenever $\mathbb{B}' \subset \mathbb{B}''$ are finite subsets of $\mathbb{B}$. Indeed, one obtains $\mu_{\mathbb{B}'}^{\boldsymbol{\eta}^{(v)}}$ from $\mu_{\mathbb{B}''}^{\boldsymbol{\eta}^{(v)}}$ by the conditioning on the decreasing event $\eta_b = 0$ for all $b \in \mathbb{B}'' \setminus \mathbb{B}'$. In the same way we get $\mu_{\mathbb{B}'}^{\boldsymbol{\eta}^{(o)}} \geqslant_{\mathrm{FKG}} \mu_{\mathbb{B}''}^{\boldsymbol{\eta}^{(o)}}$. Using these inequalities it is easy to prove the existence of the limits $\mu_{\mathbb{B}}^{\boldsymbol{\eta}^{(v)}}$ and $\mu_{\mathbb{B}}^{\boldsymbol{\eta}^{(o)}}$.

Due to the fact that for every bond $b = \langle x, y \rangle$ and every $\boldsymbol{\eta}$ one has

$$\mu_b^{\boldsymbol{\eta}}(\eta_b = 1) = \begin{cases} p & \text{if } x \text{ is connected to } y \text{ in } \mathbb{B}(\boldsymbol{\eta}) \setminus b, \\ \dfrac{p}{p + (1-q)\, p} & \text{otherwise,} \end{cases} \tag{4.1}$$

and using the Holley theorem (see ref. 11, Theorem 4.8) we get

$$\mu_{\mathbb{B}'}^{\boldsymbol{\eta}^{(v)}} \underset{\mathrm{FKG}}{\leqslant} \mu_{\mathbb{B}'}^{\boldsymbol{\eta}} \underset{\mathrm{FKG}}{\leqslant} \mu_{\mathbb{B}'}^{\boldsymbol{\eta}^{(o)}}. \tag{4.2}$$

Hence, to prove the existence of $\mu_{\mathbb{B}}^{\boldsymbol{\eta}}$ and its independence of $\boldsymbol{\eta}_{\mathbb{B}}$ it is sufficient to show that $\mu_{\mathbb{B}}^{\boldsymbol{\eta}^{(o)}} = \mu_{\mathbb{B}}^{\boldsymbol{\eta}^{(v)}}$.

Consider now the event D_m, which is realised if there is no occupied vertical bond in $\mathbb{B}$ between the planes $x_d = m$ and $x_d = m+1$, $D_{m,n} = \bigcup_{i=m}^{n-1} D_i$. Take an increasing function f depending only on the bonds from $\mathbb{B}_{-m}^m$. Then for $n > m$ one gets

$$\begin{aligned} \mu_{\mathbb{B}_{-n}^n}^{\boldsymbol{\eta}^{(o)}}(f) &= \mu_{\mathbb{B}_{-n}^n}^{\boldsymbol{\eta}^{(o)}}(f \mid D_{-n,-m}^c \cup D_{n,m}^c)\, \mu_{\mathbb{B}_{-n}^n}^{\boldsymbol{\eta}^{(o)}}(D_{-n,-m}^c \cup D_{m,n}^c) \\ &\quad + \sum_{i=m}^{n-1} \sum_{j=-n}^{m-1} \mu_{\mathbb{B}_{-n}^n}^{\boldsymbol{\eta}^{(o)}}(f \mid E_{i,j})\, \mu_{\mathbb{B}_{-n}^n}^{\boldsymbol{\eta}^{(o)}}(E_{i,j}), \end{aligned} \tag{4.3}$$

where $E_{i,j} = D_i \cap D_j \cap D_{-n,-j}^c \cap D_{i+1,n}^c$.

First we estimate the probability of event $D_{m,n}$. We use the fact, that the random cluster measure $\mu_{\mathbb{B}'}$ with parameters p and $q > 1$ is FKG-dominated by Bernoulli percolation measure $\mu_{\mathbb{B}'}^*$ with parameter $P > p$, $P \neq 1$

(see ref. 6, Theorem 2.2). Due to the fact that $D_{m,n}$ is a decreasing event, one can write

$$\mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(o)}}(D_{m,n}) \geqslant \mu_{\mathbb{B}_{-n}^{n}}^{*}(D_{m,n}) = 1-[1-(1-P)^{|Q|}]^{n-m}. \tag{4.4}$$

As a result we get $\mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(o)}}(D_{-n,-m}^{c} \cup D_{m,n}^{c}) = \varepsilon(n) \to 0$ as $n \to \infty$. Since the boundary condition η is $\mathbb{B}$-good, we have $\mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(o)}}(f \mid E_{i,j}) = \mu_{\mathbb{B}_{j+1}^{i}}^{\eta^{(v)}}(f) \leqslant \mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(v)}}(f)$. Substituting these two facts into (4.3) one gets

$$\mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(o)}}(f) = \varepsilon(n)\, \|f\| + (1-\varepsilon(n))\, \mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(v)}}(f) \leqslant \varepsilon(n)\, \|f\| + \mu_{\mathbb{B}_{-n}^{n}}^{\eta^{(v)}}(f). \tag{4.5}$$

Taking the limit $n \to \infty$ in the last equation we get $\mu_{\mathbb{B}}^{\eta^{(v)}} \geqslant \mu_{\mathbb{B}}^{\eta^{(o)}}$. This together with (4.2) completes the proof. ∎

5. INTERFACE PARTITION FUNCTION

The aim of this section is to rewrite the partition function Z^{ξ} in terms of interfaces. Here we close follow the similar step in refs. 8 and 12 taking into account, however, the particularities or contours for random cluster model. Namely, considering the finite set $\mathbb{B}_{L,M}$ (see Theorem 2.3), let η be a configuration equal to ξ outside $\mathbb{B}_{L,M}$. It is easy to see, that the boundary of $U_{1/4}(F(\eta))$ has one infinite component $I(\eta)$, called the *interface* of η. We use I_0 to denote the interface of ξ. For any such η the interface $I(\eta)$ differs from I_0 only in the 1-neighbourhood of the set $\Lambda_{L,M}$. More precisely, $I(\eta) \setminus I_0 \subset U_1(\Lambda_{L,M})$. Let us use $\mathscr{I}_{L,M}$ to denote the set of all interfaces with this property. For any interface $I \in \mathscr{I}_{L,M}$ we define the length $\|I\|_{L,M} = |I \cap \mathbb{B}(\Lambda_{L,M})|$ and the weight $\rho_{L,M}(I) = q^{-\|I\|_{L,M}/2d}$. The interface I divides $\mathbb{R}^d$ into two open components, the upper one, $\mathbb{R}_o^d(I)$, and the lower one, $\mathbb{R}_v^d(I)$. For any $\Lambda \subset \mathbb{Z}^d$, we define $\mathbb{B}^o(\Lambda, I)$ as the set of all bonds from $\mathbb{B}(\Lambda)$ with centres within $\mathbb{R}_o^d(I)$. Similarly, we define $\mathbb{B}^v(\Lambda, I)$, $\mathbb{B}_0^o(\Lambda, I)$, $\mathbb{B}_0^v(\Lambda, I)$, $\partial\mathbb{B}^o(\Lambda, I)$, and $\partial\mathbb{B}^v(\Lambda, I)$. Notice that, say, $\mathbb{B}_0(\Lambda \cap \mathbb{R}_o^d(I)) \supset \mathbb{B}_0^o(\Lambda, I)$, $\partial\mathbb{B}(\Lambda \cap \mathbb{R}_o^d(I)) \supset \partial\mathbb{B}^o(\Lambda, I)$, and $\partial\mathbb{B}(\Lambda \cap \mathbb{R}_o^d(I)) \setminus \partial\mathbb{B}^o(\Lambda, I)$ is the set of bonds from $\mathbb{B}(\Lambda)$ intersecting the interface I.

Except the interface, all other components of the boundary of the set $U_{1/4}(F(\eta))$ are contours. We use $\mathscr{D}_{L,M}(I)$ for the set of collections of all contours of configurations that equal to ξ outside $\mathbb{B}_{L,M}$ and contain the interface I. It is not difficult to see that a configuration η that equals to ξ outside $\mathbb{B}_{L,M}$ is determined by its interface I and a contour configuration ∂ from $\mathscr{D}_{L,M}(I)$. We write $\eta(\partial, I)$ for this configuration.

It will be useful to use the shorthand $x \in C$ whenever $x \in \bigcup_{\gamma \in C} \gamma$. Similarly, we use $C \cap I$ for the set of points $(\bigcup_{\gamma \in C} \gamma) \cap I$ and $C \cap \mathbb{B}$, with

$\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)$, to denote the set of all intersections of all contours γ from C with bonds from $\mathbb{B}$. An important consequence of the convergence (3.20) is the explicit expression for the limits

$$\lim_{|\Lambda| \to \infty} \frac{\log \mathscr{Z}(\mathscr{K}_\alpha(\Lambda), \Phi_\alpha)}{|\mathbb{B}(\Lambda)|} = 2 \sum_{\substack{C \subset \mathscr{K}_\alpha \\ C \ni x}} \frac{\Phi_\alpha^T(C)}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \equiv p(\Phi_\alpha), \tag{5.1}$$

where x is an arbitrary point, that can be written as $x = \frac{1}{4} y + \frac{3}{4} z$ with $\langle y, z \rangle \in \mathbb{B}(\mathbb{Z}^d)$. Moreover, a standard consequence of the Pirogov–Sinai theory[10] is that for $p = p_c$, where both phases v and o are stable, one has

$$e_v - 2 \sum_{\substack{C \subset \mathscr{K}_v \\ C \ni x}} \frac{\Phi_v^T(C)}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} = e_o - 2 \sum_{\substack{C \subset \mathscr{K}_o \\ C \ni x}} \frac{\Phi_o^T(C)}{|C \cap \mathbb{B}(\mathbb{Z}^d)|}. \tag{5.2}$$

For $\alpha \in \{o, v\}$, let $\mathscr{K}^\alpha_{L,M}(I) \subset \mathscr{K}_\alpha$ be the set of all α-contours contained in $U_1(\Lambda_{L,M}) \cap \mathbb{R}^d_\alpha(I)$ and let $\chi_\alpha \colon \exp(\mathscr{K}_\alpha) \mapsto \{0, 1\}$, be the function defined by $\chi_\alpha(C) = 1$ if there are two bonds b, b' in the same hypercube of $\mathbb{Z}^d$, b is in $\mathbb{B}^\alpha(\Lambda_{L,M}, I_0)$ and b' is in $\mathbb{B}^\alpha(\mathbb{Z}^d, I_0) \setminus \mathbb{B}^\alpha(\Lambda_{L,M}, I_0)$, such that C intersects both b and b'. Otherwise $\chi_\alpha(C) = 0$.

In the following lemma we rewrite the partition function as a sum over interfaces and, using f to denote the common value (5.2), we extract the normalisation factor that does not depend on a particular interface I.

Lemma 5.1.

$$Z^\xi_{\mathbb{B}_{L,M}} = N_{L,M} \sum_{I \in \mathscr{I}_{L,M}} Z_{L,M}(I), \tag{5.3}$$

where

$$Z_{L,M}(I) = \frac{\rho_{L,M}(I)}{\rho_{L,M}(I_0)} \exp\Bigg\{ - \sum_{\alpha = o, v} \sum_{\substack{C \subset \mathscr{K}_\alpha \\ C \cap I \neq \varnothing}} \Phi_\alpha^T(C) \Bigg[\frac{|C \cap \mathbb{B}^\alpha(\Lambda_{L,M}, I)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} - \chi_\alpha(C) \frac{|C \cap \mathbb{B}(\Lambda_{L,M})|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \Bigg] \Bigg\} \tag{5.4}$$

and $N_{L,M}$ is the normalisation factor

$$N_{L,M} = \rho_{L,M}(I_0)\, q^{1 - |\partial \mathbb{B}^v(\Lambda_{L,M}, I_0)|/2d} (1-p)^{-|\partial \mathbb{B}^v(\Lambda_{L,M}, I_0)|} \times \exp\Bigg\{ -f\, |\mathbb{B}(\Lambda)| - \sum_{\alpha = o, v} \sum_{\substack{C \subset \mathscr{K}_\alpha \\ \chi_\alpha(C) = 1}} \Phi_\alpha^T(C) \frac{|C \cap \mathbb{B}(\Lambda)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \Bigg\}. \tag{5.5}$$

Proof. Applying the methods similar to proof of the Lemma 3.1 we first rewrite the random-cluster partition function using contours. We have to take in account that, for any $\boldsymbol{\eta}$ equal to ξ outside $\mathbb{B}_{L,M}$, we have

$$k_{\mathbb{B}_{L,M}}(\boldsymbol{\eta}) = 1 + |\partial(\boldsymbol{\eta}) \cap \mathscr{K}_v| + E_\Lambda(\boldsymbol{\eta}) \tag{5.6}$$

and

$$\begin{aligned} 2d\,|\Lambda_{L,M} \cap \mathbb{V}(\mathbb{B}(\boldsymbol{\eta}))| &= 2\,|\mathbb{B}(\boldsymbol{\eta}) \cap \mathbb{B}_0(\Lambda_{L,M})| + 2\,|\partial_2\boldsymbol{\eta} \cap \mathbb{B}_0(\Lambda_{L,M})| \\ &\quad + |\mathbb{B}(\boldsymbol{\eta}) \cap \partial\mathbb{B}_0^o(\Lambda_{L,M}, I_0)| + |\partial_1\boldsymbol{\eta} \cap \mathbb{B}_0(\Lambda_{L,M})| \\ &\quad + |\partial_1\boldsymbol{\eta} \cap \partial\mathbb{B}_0^v(\Lambda_{L,M}, I_0)| + |\partial_2\boldsymbol{\eta} \cap \partial\mathbb{B}_0^o(\Lambda_{L,M}, I_0)|. \end{aligned}$$

For any ∂ from $\mathscr{D}_{L,M}(I)$ we have

$$\begin{aligned} \|I\|_{L,M} + \sum_{\gamma\in\partial} \|\gamma\| &= 2\,|\partial_2\boldsymbol{\eta}(\partial, I) \cap \mathbb{B}_0(\Lambda_{L,M})| + |\partial_1\boldsymbol{\eta}(\partial, I) \cap \mathbb{B}_0(\Lambda_{L,M})| \\ &\quad + |\partial_1\boldsymbol{\eta}(\partial, I) \cap \partial\mathbb{B}_0^v(\Lambda_{L,M}, I_0)| + |\partial_2\boldsymbol{\eta}(\partial, I) \cap \partial\mathbb{B}_0^o(\Lambda_{L,M}, I_0)| \\ &\quad + |\partial\mathbb{B}^o(\Lambda_{L,M}, I_0) \setminus \mathbb{B}(\boldsymbol{\eta}(\partial, I))|. \end{aligned} \tag{5.7}$$

After a straightforward computation, we get

$$\begin{aligned} Z^{\xi}_{B_{L,M}} &= q^{1-|\partial\mathbb{B}^v(\Lambda_{L,M}, I_0)|/2d}(1-p)^{-|\partial\mathbb{B}^v(\Lambda_{L,M}, I_0)|} \\ &\quad \times \sum_{I\in\mathscr{I}_{L,M}} \rho_{L,M}(I) \sum_{\partial\in\mathscr{D}_{L,M}(I)} e^{-e_o\,|\mathbb{B}(\Lambda_{L,M}) \cap \mathbb{B}(\boldsymbol{\eta}(\partial, I))|} \\ &\quad \times e^{-e_v\,|\mathbb{B}(\Lambda_{L,M}) \setminus \mathbb{B}(\boldsymbol{\eta}(\partial, I))|} \prod_{\gamma\in\partial} \rho(\gamma). \end{aligned} \tag{5.8}$$

Now, we apply the iteration procedure used in the proof of Lemma 3.2 separately for upper and lower part of $\mathbb{B}_{L,M}$. One has to observe, that all external contours above (resp. below) the interface are o-contours (resp. v-contours). As a result, we get

$$\begin{aligned} Z^{\xi}_{L,M} &= \sum_{I\in\mathscr{I}_{L,M}} q^{1-|\partial\mathbb{B}^v(\Lambda_{L,M}, I_0)|/2d}(1-p)^{-|\partial\mathbb{B}^v(\Lambda_{L,M}, I_0)|} \\ &\quad \times e^{-e_o\,|\mathbb{B}^o(\Lambda_{L,M}, I)|} e^{-e_v\,|\mathbb{B}^v(\Lambda_{L,M}, I)|} \\ &\quad \times \rho_{L,M}(I)\, \mathscr{Z}(\mathscr{K}^o_{L,M}(I), \Phi_o)\, \mathscr{Z}(\mathscr{K}^v_{L,M}(I), \Phi_v). \end{aligned} \tag{5.9}$$

The required result follows then by substituting

$$\exp\left[p(\Phi_\alpha)\, |\mathbb{B}^\alpha(\Lambda_{L,M}, I)| - \sum_{\substack{C \subset \mathscr{K}_\alpha \\ C \cap \mathbb{B}^\alpha(\Lambda_{L,M}, I)^c \neq \varnothing}} \Phi_\alpha^T(C)\, \frac{|C \cap \mathbb{B}^\alpha(\Lambda_{L,M}, I)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \right] \tag{5.10}$$

for $\mathscr{Z}(\mathscr{K}^\alpha_{L,M}(I), \Phi_\alpha)$ according to (3.19) and (5.1) (see also Theorem 2.2) and taking into account that $f = e_v - p(\Phi_v) = e_o - p(\Phi_o)$. ∎

6. WALLS

In the preceding section we introduced interfaces in the box $\Lambda_{L,M}$. An obvious generalisation of the notion of an interface is to consider any configuration $\boldsymbol{\eta} \in \Omega$ with a single infinite component I of the boundary of $U_{1/4}(F(\boldsymbol{\eta}))$ such that the set $I \setminus I_0$ has only finite components. We write $\mathscr{I}$ for the set of all interfaces in this sense and use $\mathscr{I}_L$ to denote the set of all interfaces differing from I_0 in the 1-neighbourhood of a cylinder $\Lambda_L = \Lambda_{L,\infty} = ([-L, L] \cap \mathbb{Z})^{d-1} \times \mathbb{Z}$.

Every interface $I \in \mathscr{I}$ can be characterised by specifying its irregularities with respect to I_0. We will call these irregularities walls. The precise definition follows. First, we introduce vertical shifts T_h: $\mathbb{R}^d \mapsto \mathbb{R}^d$, $(x_1,\ldots, x_d) \mapsto (x_1,\ldots, x_d + h)$. For any set $\Delta \subset \mathbb{R}^d$ we define the cylinder $C(\Delta) = \bigcup_{h \in \mathbb{R}} T_h(\Delta)$ and the "projection" of Δ on I_0, $\pi(\Delta) = C(\Delta) \cap I_0$, $\pi^{-1}(\pi(\Delta)) = C(\Delta)$. Every interface can be divided into two disjoint subsets $I = G(I) \cup B(I)$, its good and bad part:

$$G(I) = \{x \in I : C(x) \cap I = x\}; \quad B(I) = I \setminus G(I). \tag{6.1}$$

Connected components of the closed set $\overline{U_{1/2}(B(I))} \cap I$ are called the *walls* of I.

On the other hand, a set $w \subset \mathbb{R}^d$ is called a wall, if there exists a box $\Lambda_{L,M}$ and an interface $I \in \mathscr{I}_{L,M}$ such that w is a wall of I. Moreover, a wall w is called a *standard wall*, if there exists an interface I such that w is the only wall of I. Let $\mathscr{W}$ be the set of all standard walls. We say, that two walls w_1 and w_2 are compatible, if the intersection of their projections $\pi(w_1)$, $\pi(w_2)$ is empty. We use $\mathscr{E}$ to denote the set of all compatible families of standard walls.

For any standard wall w we define $\operatorname{Ext} w$ as the only infinite component of $I_0 \setminus \pi(w)$ and $\operatorname{Int} w = I_0 \setminus (\pi(w) \cup \operatorname{Ext} w)$. We say that a wall w is inside of a wall $\bar{w}$, if $\pi(w) \subset \operatorname{Int} \bar{w}$. A compatible collection $\mathbb{W}$ of walls is called admissible, if every wall from $\mathbb{W}$ is inside of finitely many walls from $\mathbb{W}$. Let $\mathscr{E}^a$ be the set of all admissible collections of walls.

The following obvious geometrical lemma describes the mutual relations between walls, standard walls and interfaces.

Lemma 6.1. (a) For every wall w there exits one and only one $h = h(w)$ such that $T_h(w)$ is a standard wall. We call the wall $T_{h(w)}(w)$ the standard position of w.

(b) The mapping $\mathbb{W}(\cdot)$ that assigns to any interface I the collection of its walls in standard positions maps $\mathscr{I}$ into $\mathscr{E}$ and is one-to-one from the set $\mathscr{I}^a = \mathbb{W}^{-1}(\mathscr{E}^a)$ to $\mathscr{E}^a$.

Proof. See Appendix A of ref. 8. ▮

For every standard wall w we define its length by $\|w\| = |w \cap \mathbb{B}(\mathbb{Z}^d)|$ and its energy

$$E(w) = \frac{\log q}{2d} (\|w\| - \|\pi(w)\|). \tag{6.2}$$

Note, that there exists constant $c_w = c_w(d)$ depending only on dimension, such that $1 > c_w > 0$ and $\|w\| \geqslant (1 - c_w)^{-1} \|\pi(w)\|$. Hence, we have

$$E(w) \geqslant c_w \frac{\log q}{2d} \|w\|. \tag{6.3}$$

It is now easy to rewrite the partition function $Z_{L,M}(I)$ in terms of walls.

Lemma 6.2.

$$Z_{L,M}(I) = \prod_{w \in \mathbb{W}(I)} \exp[-E(w)] \exp\Bigg\{ - \sum_{\alpha = o, v} \sum_{\substack{C \subset K_\alpha \\ C \cap I \neq \varnothing}} \Phi_\alpha^T(C) \\ \times \left[\frac{|C \cap \mathbb{B}^\alpha(\Lambda_{L,M}, I)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} - \chi_\alpha(C) \frac{|C \cap \mathbb{B}(\Lambda_{L,M})|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \right] \Bigg\} \tag{6.4}$$

Proof. According to the definition of $\rho_{L,M}(I)$ we have

$$\log \frac{\rho_{L,M}(I)}{\rho_{L,M}(I_0))} = -(\|I\|_{L,M} - \|I_0\|_{L,M}) \frac{\log q}{2d}. \tag{6.5}$$

However, it is easy to see from the definition of the wall that

$$\|I\|_{L,M} = \|I_0\|_{L,M} + \sum_{w \in \mathbb{W}(I)} (\|w\| - \|\pi(w)\|). \tag{6.6}$$

Combining these facts with (6.2) and Lemma 5.1 completes the proof. ▮

We will now investigate the random cluster model in a cylinder Λ_L under the boundary condition ξ. We use $\mathbb{B}_L$ to denote $\mathbb{B}_{L,\infty}$. Since the configuration ξ is evidently a $\mathbb{B}_L$-good boundary condition, we know from Proposition 4.1, that there is a unique limit random cluster measure

$$\mu^{\xi}_{\mathbb{B}_L} = \lim_{M \to \infty} \mu^{\xi}_{\mathbb{B}_{L,M}}. \tag{6.7}$$

First, we have to verify that there exists $\mu^{\xi}_{\mathbb{B}_L}$-a.s. an interface. It is not a priory clear, because we have defined that the interface has only finite components of $I \setminus I_0$.

Lemma 6.3. There exists $\mu^{\xi}_{\mathbb{B}_L}$-a.s. an interface.

Proof. Let $\mathscr{I}^{(n)}_L$ be the set $\{\eta : \eta = \xi$ ouside $\mathbb{B}_L$ and $I(\eta) \subset \mathbb{R}^{d-1} \times [-n, n]\}$. The assertion of the lemma is an easy consequence of the fact

$$\lim_{n \to \infty} \mu^{\xi}_{\mathbb{B}_L}(\mathscr{I}^{(n)}_L) = 1. \tag{6.8}$$

To prove this we chose $\varepsilon > 0$, and we define the events D_m, which is realised if all vertical bonds in $\mathbb{B}_L$ between the planes $x_d = m$ and $x_d = m+1$ are vacant, and E_m, which is realised if all horizontal bonds in the plane $x_d = m$ are occupied. It is easy to see

$$\mathscr{I}^{(n)}_L \supset \left(\bigcup_{0 < k \leqslant n} D_{-k} \right) \cap \left(\bigcup_{0 < l \leqslant n} E_l \right). \tag{6.9}$$

Since $\mu^{\xi}_{\mathbb{B}_L}$ is FKG-dominated by percolation measure $\mu^{\star}_{\mathbb{B}_L}$ with parameter $1 > P \geqslant p_c$, and D_k are decreasing events, we have for n large enough

$$\mu^{\xi}_{\mathbb{B}_L}\left(\bigcup D_{-k} \right) \geqslant \mu^{\star}_{\mathbb{B}_L}\left(\bigcup D_{-k} \right) = 1 - (1 - (1-P)^{(2L+1)^{d-1}})^n \geqslant 1 - \varepsilon/2. \tag{6.10}$$

On the other side, $\mu^{\xi}_{\mathbb{B}_L}$ FKG-dominates the percolation measure $\mu^{\star}_{\mathbb{B}_L}$ with parameter P, such that $\frac{p}{q(1-p)}=\frac{P}{1-P}$ (see ref. 6, Theorem 4.8). Since E_l are increasing events, we have for n large enough

$$\mu^{\xi}_{\mathbb{B}_L}\left(\bigcup E_l\right)\geqslant \mu^{\star}_{\mathbb{B}_L}\left(\bigcup E_l\right)=1-(1-P^{KL^{d-1}})^n\geqslant 1-\varepsilon/2, \tag{6.11}$$

where $K=K(d)$ is a positive constant. Since ε was arbitrary, (6.8) follows from (6.9)–(6.11). ∎

Lemma 6.4. Let q be large enough. Then

(a) For every $L\in\mathbb{N}$ there exists a mapping $K_L\colon \mathscr{I}_L\mapsto(0,\infty)$ such that $\sum_{I\in\mathscr{I}_L}K_L(I)<\infty$ and $Z_{L,M}(I)\leqslant K_L(I)$ for every $M\in\mathbb{N}$ and $I\in\mathscr{I}_{L,M}$.

(b) There exists a finite limit $\lim_{M\to\infty}Z_{L,M}(I)$ and it equals

$$Z_L(I)=\prod_{w\in\mathbb{W}(I)}\exp[-E(w)]\times\exp\Bigg\{-\sum_{\alpha=o,v}\sum_{\substack{C\subset\mathscr{K}_\alpha\\ C\cap I\neq\varnothing}}\Phi^T_\alpha(C)\Bigg[\frac{|C\cap\mathbb{B}^\alpha(\Lambda_L,I)|}{|C\cap\mathbb{B}(\mathbb{Z}^d)|}$$
$$-\chi_\alpha(C)\frac{|C\cap\mathbb{B}(\Lambda_L)|}{|C\cap\mathbb{B}(\mathbb{Z}^d)|}\Bigg]\Bigg\} \tag{6.12}$$

(c) The probability $P^{\mathscr{I}}_{L,M}$ on $\mathscr{I}_{L,M}$ defined, for $I\in\mathscr{I}_{L,M}$, by

$$P^{\mathscr{I}}_{L,M}(I)=\mu^{\xi}_{\mathbb{B}_{L,M}}(\{\eta: I(\eta)=I\})=\frac{Z_{L,M}(I)}{\sum_{I'\in\mathscr{I}_{L,M}}Z_{L,M}(I')} \tag{6.13}$$

converges to a probability $P^{\mathscr{I}}_L$ on $\mathscr{I}_L$ that is proportional to Z_L, i.e.

$$P^{\mathscr{I}}_L(I)=\frac{Z_L(I)}{\sum_{I'\in\mathscr{I}_L}Z_L(I')}. \tag{6.14}$$

Proof. We start by bounding the absolute value of the sum in the exponent in (6.4). We divide it into two parts. First, we take all clusters intersecting the interface inside of $U_1(\Lambda_{L,M})$. To use expression (3.20), we have to chose an appropriate finite set of points on the interface. Thus, let us first introduce the set $\mathbb{Z}^d_{1/2}=\frac{1}{2}\mathbb{Z}^d+(\frac{1}{4},\dots,\frac{1}{4})$. It is clear, that if a cluster intersects the interface, their intersection has to contain at least one point from $\mathbb{Z}^d_{1/2}$. The number of such points in $U_1(\Lambda_{L,M})$ is smaller than $c_c\,\|I\|_{L,M}$, where $c_c=c_c(d)$ is a constant depending only on the dimension. It is easy to see that $c_c(d)\leqslant 2^{d+1}$. Using this we can bound the first part of sum by $2c_c\,\|I\|_{L,M}$.

It remains to bound the sum over clusters intersecting the interface only outside $U_1(\Lambda_{L,M})$. From (3.20) we have

$$\sum_{\substack{C \ni x \\ \|C\| > \ell}} |\Phi^T(C)| \leqslant e^{-\omega\ell}. \tag{6.15}$$

Observing that the number of points in $\mathbb{Z}^d_{1/2} \cup I_0$ at distance of order ℓ from $\Lambda_{L,M}$ is of the order $(L+\ell)^{d-2}$, it is easy to see that the second part of the sum is convergent and can be bounded by some constant $\bar{K}_L = \bar{K}_L(d)$.

According to the definition of $E(w)$ and using the results of the preceding discussion, we can write

$$\begin{aligned} Z_{L,M} &\leqslant e^{\bar{K}_L} \exp\left\{ -\sum_{w \in \mathbb{W}(I)} \frac{\log q}{2d} (\|w\| - \|\pi(w)\|) + 2c_c \|I\|_{L,M} \right\} \\ &= e^{\bar{K}_L} \exp\left\{ -\sum_{w \in \mathbb{W}(I)} \left(\frac{\log q}{2d} - 2c_c \right) (\|w\| - \|\pi(w)\|) + 2c_c \|I_0\|_{L,M} \right\}. \end{aligned} \tag{6.16}$$

Using (6.3) and taking q large enough to be sure that $(2d)^{-1} \log q - 2c_c \geqslant 0$ we have:

$$\begin{aligned} Z_{L,M}(I) &\leqslant e^{\bar{K}_L} \exp\left\{ -c_w \sum_{w \in \mathbb{W}(I)} \left(\frac{\log q}{2d} - 2c_c \right) \|w\| + 2c_c \|I_0\|_L \right\} \\ &\equiv K_L(I). \end{aligned} \tag{6.17}$$

The sum of $K_L(I)$ can be bounded in the following way

$$\begin{aligned} \sum_{I \in \mathcal{I}_L} K_L(I) &\leqslant \exp(\bar{K}_L + 2c_c \|I_0\|_L) \\ &\quad \times \prod_{x \in I_0 \cap \mathbb{B}_{\Lambda_L}} \sum_{\substack{w \in \mathcal{W} \\ x \in w}} \exp\left[-c_w \left(\frac{\log q}{2d} - 2c_c \right) \|w\| \right]. \end{aligned} \tag{6.18}$$

The number of walls that contains an arbitrary site $x \in \mathbb{R}^d$ can be bounded in the same way as the number of such contours,

$$|\{w : w \ni x, \|w\| = n\}| \leqslant c^n. \tag{6.19}$$

Let m_w be the minimal length of wall. Then

$$\sum_{\substack{w \in \mathscr{W} \\ i \in w}} \exp\left[-c_w \|w\| \left(\frac{\log q}{2d} - 2c_c\right)\right] \leqslant \sum_{n=m_w}^{\infty} c^n \exp\left[-nc_w \left(\frac{\log q}{2d} - 2c_c\right)\right]$$
$$\leqslant \frac{\{c \exp[-c_w(\log q/2d - 2c_c)]\}^{m_w}}{1 - c \exp[-c_w(\log q/2d - 2c_c)]} \leqslant 1 \tag{6.20}$$

if

$$c_w \left(\frac{\log q}{2d} - 2c_c\right) \geqslant \log(2c). \tag{6.21}$$

Proof of (b) follows from the facts that $|\Phi_\alpha^T(C)|$ decreases exponentially with the size of C and, for any finite C, we have $|C \cap \mathbb{B}^\alpha(\Lambda_{L,M}, I)| \to |C \cap \mathbb{B}^\alpha(\Lambda_L, I)|$ and $|C \cap \mathbb{B}(\Lambda_{L,M})| \to |C \cap B(\Lambda_L)|$ as $M \to \infty$. The statement (c) is a consequence of (a). ∎

7. PROOF OF THEOREM 2.3(I)–(III)

For every finite, compatible collection of walls $\mathbb{W}$ we use $\mathscr{M}_{\mathbb{W}}$ to denote the set $\mathscr{M}_{\mathbb{W}} = \{\bar{\mathbb{W}}: \bar{\mathbb{W}} \in \mathscr{E}, \mathbb{W} \subset \bar{\mathbb{W}}\}$ and put $\|\mathbb{W}\| = \sum_{w \in \mathbb{W}} \|w\|$.

We will prove the following lemma in the Appendix A.

Lemma 7.1. Let q large enough. Then we have

(a) There is $\bar{c} > 0$, such that

$$P_L^{\mathscr{I}}(\mathscr{M}_{\mathbb{W}}) \leqslant \exp[-\bar{c}\, \|\mathbb{W}\|] \tag{7.1}$$

for any finite collection of walls $\mathbb{W} \in \mathscr{E}_L$.

(b) The probability measures $P_L^{\mathscr{I}}$ converge weakly to the measure $P^{\mathscr{I}}$ on $\mathscr{I}$.

(c) The limiting measure $P^{\mathscr{I}}$ satisfies $P^{\mathscr{I}}(\mathscr{I}^a) = 1$.

Using the preceding lemma we will now prove the existence of the limiting state μ^ξ (the claim (i) of Theorem 2.3). Actually, we will prove even more: an explicit expression for μ^ξ in terms of the limiting measure $P^{\mathscr{I}}$ that allows to verify easily the claims (ii) and (iii) of Theorem 2.3. To this end, let us first introduce the measure $\mu(\cdot \mid I)$ on $\{0, 1\}^{\mathbb{B}(\mathbb{Z}^d)}$ defined for any $I \in \mathscr{I}$ as follows. Let $\mathbb{B}_I$ be the set of all bonds whose value is fixed by existence of the interface I. Namely, $\mathbb{B}_I$ is defined as the set of all bonds $b \in \mathbb{B}(\mathbb{Z}^d)$ that either intersect I or lie on the boundary of $U_{1/4}(I)$. For any

$\Lambda \subset \mathbb{Z}^d$ (possibly $\Lambda = \mathbb{Z}^d$), the measures $\mu^o_{\Lambda,I}(\cdot) = \lim_{\mathbb{B} \to \mathbb{B}^o(\Lambda,I) \setminus \mathbb{B}_I} \mu^{\eta^o}_{\mathbb{B}}(\cdot)$ on $\{0,1\}^{\mathbb{B}^o(\Lambda,I) \setminus \mathbb{B}_I}$ are well defined by the same argument as in (2.5). Similarly for $\mu^v_{\Lambda,I}$. Introducing also the Dirac measure $\delta_{\Lambda,I}$ on $\{0,1\}^{\mathbb{B}_I \cap \mathbb{B}(\Lambda)}$ defined by $\delta_I(\boldsymbol{\eta}) = 1$, if $\eta(b) = 0$ for all $b \in \mathbb{B}_I \cap \mathbb{B}(\Lambda)$ that intersect I and $\eta(b) = 1$ for all $b \in \mathbb{B}_I \cap \mathbb{B}(\Lambda)$ that lie on the boundary of $U_{1/4}(I)$, and $\delta_I(\boldsymbol{\eta}) = 0$ otherwise, we *define*

$$\mu_\Lambda(\cdot \mid I) = \mu^v_{\Lambda,I} \otimes \delta_{\Lambda,I} \otimes \mu^o_{\Lambda,I}(\cdot) \tag{7.2}$$

and denote $\mu(\cdot \mid I) = \mu_{\mathbb{Z}^d}(\cdot \mid I)$ and $\mu_L(\cdot \mid I) = \mu_{\Lambda_L}(\cdot \mid I)$.

Let ϕ be a cylindrical function living on the B_{L_0,M_0}. For finite volume $\Lambda_{L,M}$, $L \geqslant L_0$, $M \geqslant M_0$, we clearly have

$$\mu^{\xi}_{\mathbb{B}_{L,M}}(\phi) = \int \mu_{\Lambda_{L,M}}(\phi \mid I)\, dP^{\mathscr{I}}_{L,M}(I) \tag{7.3}$$

(cf. (6.13)). Recalling that with respect to the limit $\mu^{\xi}_{\mathbb{B}_L}$ an interface exists a.s., observing that

$$\lim_{M \to \infty} \mu_{\Lambda_{L,M}}(\phi \mid I) = \mu_L(\phi \mid I), \tag{7.4}$$

and using (6.8), we also have

$$\mu^{\xi}_{\mathbb{B}_L}(\phi) = \int \mu_L(\phi \mid I)\, dP^{\mathscr{I}}_L(I). \tag{7.5}$$

Next, we will show that

$$\lim_{L \to \infty} \int \mu_L(\phi \mid I)\, dP^{\mathscr{I}}_L(I) = \int \mu(\phi \mid I)\, dP^{\mathscr{I}}(I), \tag{7.6}$$

yielding thus the existence of the limiting μ^{ξ} as well as the expression

$$\mu^{\xi}(\phi) = \int \mu(\phi \mid I)\, dP^{\mathscr{I}}(I). \tag{7.7}$$

Using the convergence $\lim_{L \to \infty} \mu_L(\phi \mid I) = \mu(\phi \mid I)$, i.e.

$$\int |\mu_L(\phi \mid I) - \mu(\phi \mid I)|\, dP^{\mathscr{I}}_L(I) \leqslant \frac{\varepsilon}{2} \tag{7.8}$$

for L large, for proving

$$\left| \int \mu_L(\phi \mid I)\, dP^{\mathscr{I}}_L(I) - \int \mu(\phi \mid I)\, dP^{\mathscr{I}}(I) \right| \leqslant \varepsilon, \tag{7.9}$$

it suffices to show that

$$\left| \int \mu(\phi \mid I)\, dP_L^{\mathscr{I}}(I) - \int \mu(\phi \mid I)\, dP^{\mathscr{I}}(I) \right| \leqslant \frac{\varepsilon}{2} \tag{7.10}$$

for large L. Let us introduce the function

$$f_k(I) = \chi_{L_0}^k(I)\, \mu(\phi \mid I) \tag{7.11}$$

for every $k \in \mathbb{N} \cup \infty$, where $\chi_{L_0}^k(I) = 1$, if for every wall from $\mathbb{W}(I)$ that intersects $\pi^{-1}(B_{L_0, M_0})$ holds $\|w\| \leqslant k$, and $\chi_{L_0}^k(I) = 0$ otherwise. Using Lemma 7.1 (a) it is easy to see, that the limit

$$\lim_{k \to \infty} \int f_k(I)\, dP_L^{\mathscr{I}}(I) \tag{7.12}$$

exists and is equal to

$$\int f_\infty(I)\, dP_L^{\mathscr{I}}(I) = \int \mu_L(\phi \mid I)\, dP_L^{\mathscr{I}}(I) \tag{7.13}$$

for every L large enough.

For any sequence of interfaces $I_n \in \mathscr{I}^a$ converging to $I \in \mathscr{I}$ (in the sense that the distance between the symmetrical difference $I \triangle I_n$ and the origin goes to infinity) we have $f_k(I_n) \to f_k(I)$. It is an easy consequence of the fact that $\chi_{L_0}^k(I)$ has finite support and of the limiting properties of $\mu(\phi \mid I)$ again. Hence, f_k is continuous and according to the weak convergence of $P_U^{\mathscr{I}}$ demonstrated in Lemma 7.1 we have

$$\left| \int f_k(I)\, dP_L^{\mathscr{I}}(I) - \int f_k(I)\, dP^{\mathscr{I}}(I) \right| \leqslant \frac{\varepsilon}{2}. \tag{7.14}$$

The existence of limit random-cluster measure comes from (7.8)–(7.14).

The assertion (ii) and (iii) of Theorem 2.3 are consequences of existence of interface and the properties of $\mu(\phi \mid I)$. ∎

8. PROOF OF THEOREM 2.3(IV)

We use the approach from refs. 6, 7, and 14 to prove that μ^ξ is a Gibbs random cluster measure. First we state that the specifications $\mu_{\mathbb{B}}^\xi$ are "almost surely quasilocal" in the language of ref. 13. For finite sets Λ, Δ with $\Lambda \subset \Delta \subset \mathbb{Z}^d$, let $\mathcal{M}_{\Delta,\Lambda}$ be the event

$$\mathcal{M}_{\Delta,\Lambda} = \{\boldsymbol{\eta} \mid \forall x, y \in \Lambda \; x \leftrightarrow \Delta^c \text{ and } y \leftrightarrow \Delta^c \text{ implies } x \leftrightarrow y \text{ in } \mathbb{B}_0(\Delta)\} \tag{8.1}$$

Lemma 8.1. (i) Let $\mathbb{B} \subset \mathbb{B}(\mathbb{Z}^d)$ be a finite set, and let f be a cylinder function depending only on the bonds from $\mathbb{B}$. Then function

$$\boldsymbol{\eta} \mapsto \mathbb{1}_{\mathcal{M}_{\Delta,\Lambda}}(\boldsymbol{\eta})\, \mu^{\boldsymbol{\eta}}_{\mathbb{B}} \tag{8.2}$$

is quasilocal for any pair of finite sets Δ, Λ with $\Delta \supset \Lambda \subset \mathbb{V}(\mathbb{B})$.

(ii) Let μ is a random cluster limit measure with at most one infinite cluster and $\Lambda \subset \mathbb{Z}^d$ finite. Then

$$\mu(\mathcal{M}_{\Delta,\Lambda}) \nearrow 1 \qquad \text{as} \quad \Delta \nearrow \mathbb{Z}^d. \tag{8.3}$$

Proof. (i) Recalling the definition (2.1) of $\mu^{\boldsymbol{\eta}}_{\mathbb{B}}$, it is sufficient to prove that the function $\boldsymbol{\eta} \mapsto \mathbb{1}_{\mathcal{M}_{\Delta,\Lambda}} q^{k_{\mathbb{B}}(\bar{\boldsymbol{\eta}}_{\mathbb{B}} \circ \boldsymbol{\eta}_{\mathbb{B}^c})}$ is quasilocal for all $\bar{\boldsymbol{\eta}}$. Let $\tilde{\Delta} \supset \Delta$, and let $\boldsymbol{\eta}$, $\boldsymbol{\eta}'$ be two configurations differing at single bond $b \in \mathbb{B}(\tilde{\Delta})^c$, $\eta_b = 0$, $\eta'_b = 1$. Suppose that $\boldsymbol{\eta} \in \mathcal{M}_{\Delta,\Lambda}$. By definition of $\mathcal{M}_{\Delta,\Lambda}$, there is only one cluster that connects Λ with Δ^c, that is why changing the state of the bond b does not affect the number $k_{\mathbb{B}}(\bar{\boldsymbol{\eta}}_{\mathbb{B}} \circ \boldsymbol{\eta}_{\mathbb{B}^c})$. This proves quasilocality of $\boldsymbol{\eta} \mapsto \mathbb{1}_{\mathcal{M}_{\Delta,\Lambda}}(\boldsymbol{\eta})\, \mu^{\boldsymbol{\eta}}_{\mathbb{B}}$ as required.

(ii) Since $\mathcal{M}_{\Delta,\Lambda} \nearrow \mathcal{M}_{\Lambda}$, where $\mathcal{M}_{\Lambda}$ is the event that there is at most one infinite component intersecting Λ, we have that $\mu(\mathcal{M}_{\Delta,\Lambda}) \nearrow \mu(\mathcal{M}_{\Lambda}) = 1$, by the assumption of that there is μ-a.s. at most one infinite cluster. ∎

We proceed with the proof of part (iv) of Theorem 2.3. Let $\mathbb{B}$ be a finite set of bonds, and let f be bounded function depending only on the bonds from $\mathbb{B}$. Since both f and $\mathbb{1}_{\mathcal{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\cdot)\, \mu^{\cdot}_{\mathbb{B}}(f)$ are quasilocal for all $\Delta \supset \mathbb{V}(\mathbb{B})$, we have

$$\mu^{\xi}(\mathbb{1}_{\mathcal{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\cdot)\, \mu^{\cdot}_{\mathbb{B}}(f)) = \lim_{L, M \to \infty} \mu^{\xi}_{\mathbb{B}_{L,M}}(\mathbb{1}_{\mathcal{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\cdot)\, \mu^{\cdot}_{\mathbb{B}}(f)) \tag{8.4}$$

and

$$\mu^{\xi}(f) = \lim_{L, M \to \infty} \mu^{\xi}_{\mathbb{B}_{L,M}}(f) = \lim_{L, M \to \infty} \mu^{\xi}_{\mathbb{B}_{L,M}}(\mu^{\cdot}_{\mathbb{B}}(f)). \tag{8.5}$$

In the last equality we have used the DLR condition for the finite volume measure $\mu^{\xi}_{\mathbb{B}_{L,M}}$, that is not difficult to check.

Let $\varepsilon > 0$. By part (ii) of Theorem 2.3 there is μ^{ξ}-a.s a unique infinite cluster, so we can use part (ii) of Lemma 8.1. Considering the boundedness of $\mu^{\cdot}_{\mathbb{B}}(f)$, we can choose Δ_1, Δ_2 and L_0, M_0, such that

$$|\mu^{\xi}(\mu^{\cdot}_{\mathbb{B}}(f)) - \mu^{\xi}(\mathbb{1}_{\mathscr{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\cdot)\, \mu^{\cdot}_{\mathbb{B}}(f))| \leqslant \frac{\varepsilon}{3} \tag{8.6}$$

and

$$|\mu^{\xi}_{\mathbb{B}_{L,M}}(\mu^{\cdot}_{\mathbb{B}}(f)) - \mu^{\xi}_{\mathbb{B}_{L,M}}(\mathbb{1}_{\mathscr{M}_{\Delta,\mathbb{V}(\mathbb{B})}}(\cdot)\, \mu^{\cdot}_{\mathbb{B}}(f))| \leqslant \frac{\varepsilon}{3} \tag{8.7}$$

provided $\Delta_1 \subset \Delta \subset \Delta_2$ and $L \geqslant L_0$, $M \geqslant M_0$. Combining (8.4)–(8.7), we get

$$|\mu^{\xi}(f) - \mu^{\xi}(\mu^{\cdot}_{\mathbb{B}}(f))| \leqslant \varepsilon \tag{8.8}$$

Since ε was arbitrary, we have $\mu^{\xi}(f) = \mu^{\xi}(\mu^{\cdot}_{\mathbb{B}}(f))$. ∎

APPENDIX A: PROOF OF LEMMA 7.1

In this appendix we prove Lemma 7.1 using the method from ref. 8.

A.1. Aggregates

The aim of this part is to rewrite the formula (6.12) for $Z_L(I)$ in terms of abstract polymer model. The proper definition of polymers of this model is not very easy and it appears naturally during the rewriting. That is why we start by simplifying the formula (6.12) and at the end of this section we state the result as a lemma.

We introduce an additional notation to simplify the expressions. Namely, we define the sets $\mathscr{C}^{\alpha} = \{C \subset \mathscr{K}_{\alpha} : C$ cannot be decomposed into two subsets C_1 and C_2 such that every $\gamma_1 \in C_1$ is compatible with every $\gamma_2 \in C_2\}$. We define the set $\mathscr{C}^{\alpha}_L$ similarly, the only difference is that $C \subset \mathscr{K}_{\alpha}(\Lambda_L)$. The standard result of the theory of cluster expansions is that $\Psi^T_{\alpha}(C) = 0$ for all $C \notin \mathscr{C}^{\alpha}$. Thus, we can replace the sums over all subsets of $\mathscr{K}_{\alpha}$ by sums over $\mathscr{C}^{\alpha}$.

We use the following observation, valid for any countable set N and any absolutely summable sequence a_n,

$$\exp\left(\sum_{n \in N} a_n\right) = \prod_{n \in N} [(\exp a_n - 1) + 1] = \sum_{\substack{K \subset N \\ \text{finite}}} \prod_{n \in K} (\exp a_n - 1). \tag{A.1}$$

The assumption about absolute summability holds true for the expression in (6.12), because

$$\sum_{\alpha} \sum_{\substack{C \in \mathscr{C}^\alpha \\ C \cap I \neq \varnothing}} \left| \Phi_\alpha^T(C) \left[\frac{|C \cap \mathbb{B}^\alpha(\Lambda_L, I)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} - \chi_\alpha(C) \frac{|C \cap \mathbb{B}(\Lambda_L)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \right] \right| \leqslant 2c_c \|I\|_L + \bar{K}, \tag{A.2}$$

as we have shown in the proof of Lemma 6.4. Defining

$$f_{L,I}^\alpha(C) = \exp\left\{ -\Phi_\alpha^T(C) \left[\frac{|C \cap \mathbb{B}^\alpha(\Lambda_L, I)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} - \chi_\alpha(C) \frac{|C \cap \mathbb{B}(\Lambda_L)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \right] \right\} - 1, \tag{A.3}$$

for every $C \in \mathscr{C}^\alpha$, we get

$$Z_L(I) = \prod_{w \in \mathbb{W}(I)} e^{-E(w)} \sum_{T^o, T^v} \prod_\alpha \prod_{C \in T_\alpha} f_{L,I}^\alpha(C). \tag{A.4}$$

Here the sum runs over all finite sets $T^\alpha \subset \mathscr{C}^\alpha$, such that every element of T^α intersects I.

We now decompose the union of sets $\mathbb{W}(I)$, T^o and T^v into disjoint components $A = (a_w, a_o, a_v)$, where $a_w \subset \mathbb{W}(I)$, $a_\alpha \subset T^\alpha$, such that

$$\pi(A) = \pi\left(\bigcup_{w \in a_w} w \cup \bigcup_{C \in a_d} C \cup \bigcup_{C \in a_o} C \right) \tag{A.5}$$

is a maximal connected component of $\pi(\mathbb{W}(I) \cup T^o \cup T^v)$. We call such A an *aggregate* of I, T^o and T^v. On the other side, the triplet (a_w, a_o, a_v) is an aggregate if it is an aggregate of any I, T^o and T^v. We say that A is a *standard aggregate* if there is an interface I such that $a_w = \mathbb{W}(I)$. Similarly as for walls, it is possible to prove that for every aggregate A there is one and only one $h = h(A)$ such that the shift $T_{h(A)}(A)$ is a standard aggregate. We use $\mathbb{A}_L$ to denote the set of all aggregates of interfaces from $\mathscr{I}_L$ and $\mathbb{A} = \bigcup_L \mathbb{A}_L$. Note, that $A \in \mathbb{A}_L$ does not mean that $A \subset U_1(\Lambda_L)$. We say that two standard aggregates are compatible if their projections on I_0 have an empty intersection, $\mathscr{F}_L$ denotes the set of all compatible collections of aggregates from $\mathbb{A}_L$. For every $R \in \mathscr{F}_L$ we define a mapping $\mathbb{W}(R) = \bigcup_{A \in R} a_w$. We use $\|A\|$ to denote the "length" of the aggregate A,

$$\|A\| = \sum_{w \in a_w} \|w\| + \sum_{C \in a_d \cup a_o} \|C\| \tag{A.6}$$

and $\|\pi(A)\| = |\mathbb{B}(\mathbb{Z}^d) \cup \pi(A)|$ to denote the "length" of its projection.

For every aggregate we introduce the aggregate functional

$$\Psi_L(A) = \prod_{w \in a_w} e^{-E(w)} \prod_{\alpha} \prod_{C \in a_\alpha} f^{\alpha}_{L, I(a_w)}(C). \tag{A.7}$$

Since the value of $f^{\alpha}_{L,I}(C)$ depends only on the walls that are in the same aggregate as C, and since $\Phi^T_\alpha(C)$ is translation invariant we can rewrite (A.4) as

$$Z_L(I) = \sum_{\substack{R \in \mathscr{F}_L \\ \mathbb{W}(R) = \mathbb{W}(I)}} \prod_{A \in R} \Psi(A). \tag{A.8}$$

Combining this fact with Lemma 6.4 one easily gets

Lemma A.1. Let q be large enough. Then

$$Z_L := \sum_{I \in \mathscr{I}_L} Z_L(I) = \mathscr{Z}(\mathbb{A}_L, \Psi_L) \tag{A.9}$$

and

$$P^{\mathscr{I}}_L(I) = (\mathscr{Z}(\mathbb{A}_L, \Psi_L))^{-1} \sum_{\substack{S \in \mathscr{F}_L \\ \mathbb{W}(S) = \mathbb{W}(I)}} \prod_{A \in S} \Psi_L(A). \tag{A.10}$$

At this place, is useful to observe that although we have defined $f^{\alpha}_{L,I}$ and Ψ_L only for finite L, their definitions are meaningful also for $L = \infty$. We will write f^{α}_I for $f^{\alpha}_{\infty, I}$ and Ψ for Ψ_∞.

A.2. Properties of Aggregate Contour Model

In Lemma A.1 we express the normalised partition function Z_L in terms of a polymer model. To be able to apply the cluster expansion to this model, we have to verify the assumptions of Theorem B.1. This will be done in Lemma A.3. Before stating it, we will prove one auxiliary lemma.

Lemma A.2. Let q be large enough. Then for any $L \in \mathbb{N} \cup \infty$, $I \in \mathscr{I}_L$, and any $x \in \mathbb{R}^d$,

$$\sum_{\substack{C \in \mathscr{C}^\alpha \\ x \in C}} |f^{\alpha}_{L,I}(C)| \exp(\omega \, \|C\|) \leqslant 1/2, \tag{A.11}$$

whenever

$$\omega \leqslant \frac{1}{2d} \log q - \omega_0 - m_\alpha^{-1}(2+2\log 2). \tag{A.12}$$

Proof. According to (3.20) and Theorem B.2 we have

$$\sum_{\substack{C \in \mathscr{C}^\alpha \\ C \ni x}} |\Phi_\alpha^T(C)| \, e^{\tilde{\omega} \|C\|} \leqslant 1 \tag{A.13}$$

with

$$\tilde{\omega} = \frac{1}{2d} \log q - \omega_0. \tag{A.14}$$

Using the fact that $|e^u - 1| \leqslant e^v |u|$ if $|u| \leqslant v$ and that $|C \cap \mathbb{B}(\mathbb{Z}^d)| \geqslant m_\alpha$ for every cluster from $\mathscr{C}^\alpha$, we have

$$\begin{aligned}
&\sum_{\substack{C \in \mathscr{C}^\alpha \\ C \ni x}} |f^\alpha_{L,I}(C)| \exp(\omega \|C\|) \\
&\quad = \sum_{\substack{C \in \mathscr{C}^\alpha \\ C \ni x}} \left| \exp\left\{ -\Phi_\alpha^T(C) \left[\frac{|C \cap \mathbb{B}^\alpha(\Lambda_L, I)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} - \chi_\alpha(C) \frac{|C \cap \mathbb{B}(\Lambda_L)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \right] \right\} - 1 \right| \\
&\quad\quad \times \exp(\omega \|C\|) \\
&\quad \leqslant \sum_{\substack{C \in \mathscr{C}^\alpha \\ C \ni x}} 2e^2 |\Phi_\alpha^T(C)| \exp(\omega \|C\|) \\
&\quad \leqslant 2e^2 \exp[(\omega - \tilde{\omega}) m_\alpha] \leqslant 1/2. \quad \blacksquare
\end{aligned} \tag{A.15}$$

Lemma A.3. Let q be large enough. Then

$$\sum_{\substack{A \in \mathbb{A}_L \\ \pi(A) \ni x}} \exp(\|\pi(A)\| + \omega \|A\|) \, \Psi_L(A) \leqslant 1, \tag{A.16}$$

for any $x \in I_0$, whenever

$$\omega \leqslant -\zeta + \frac{c_w}{2d} \log q - \omega_0 - \max_\alpha m_\alpha^{-1}(2+\log 2) \tag{A.17}$$

and

$$\zeta = 1 + \log(3c) + c_c \log 4 + \log c_{d-1}. \tag{A.18}$$

Proof. We use $\mathscr{Q}$ to denote the set of all projections of aggregates, i.e., the set of all $Q \subset I_0$, such that there exists an aggregate A for which $\pi(A) = Q$. For every $Q \in \mathscr{Q}$ and every $a_w \in \mathscr{E}$, such that $\pi(a_w) \subset Q$, we define the set $F(a_w, Q) = \{(a_o, a_v) : A = (a_w, a_o, a_v) \in \mathbb{A} \text{ and } \pi(A) = Q\}$. Using these definitions and the relation (6.3), we get

$$\begin{aligned}
&\sum_{\substack{A \in \mathbb{A}_L \\ \pi(A) \ni x}} \exp(\|\pi(A)\| + \omega \|A\|)\, \Psi_L(A) \\
&\quad \leqslant \sum_{\substack{Q \in \mathscr{Q} \\ Q \ni x}} (\exp \|Q\|) \sum_{\substack{A \in \mathbb{A}_L \\ \pi(A) = Q}} \exp(\omega \|A\|) \prod_{w \in a_w} \exp\{-E(w)\} \\
&\quad \quad \times \prod_{\alpha} \prod_{C \in a_\alpha} f^{\alpha}_{L, I(a_w)}(C) \\
&\quad \leqslant \sum_{\substack{Q \in \mathscr{Q} \\ Q \ni x}} (\exp \|Q\|) \sum_{\substack{\pi(a_w) \subset Q \\ a_w \in \mathscr{E}_L}} \exp\left[\left(\omega - c_w \frac{\log q}{2d}\right) \|a_w\|\right] \\
&\quad \quad \times \sum_{(a_o, a_v) \in F(a_w, Q)} \prod_{\alpha} \prod_{C \in a_\alpha} |\exp(\omega \|C\|)\, f^{\alpha}_{L, I(a_w)}(C)| = (1).
\end{aligned} \tag{A.19}$$

Since $\|A\| \geqslant \|I(a_w) \cap \pi^{-1}(Q)\| \equiv \|I(a_w, Q)\|$, we have

$$\begin{aligned}
(1) &\leqslant \sum_{\substack{Q \in \mathscr{Q} \\ Q \ni x}} (\exp \|Q\|) \sum_{\substack{a_w \in \mathscr{E}_L \\ \pi(a_w) \subset Q}} \exp\left[\left(\omega - c_w \frac{\log q}{2d} + \zeta\right) \|a_w\| - \zeta \|I(a_w, Q)\|\right] \\
&\quad \times \sum_{(a_o, a_v) \in F(a_w, Q)} \prod_{\alpha} \prod_{C \in a_\alpha} |\exp[(\omega + \zeta) \|C\|]\, f^{\alpha}_{I(a_w)}(C)| = (2).
\end{aligned} \tag{A.20}$$

Using the previous lemma, the fact $\omega - (2d)^{-1} c_w \log q + \zeta \leqslant 0$, and the relation between the number of points in $I(a_w, Q) \cap \mathbb{Z}^d_{1/2}$ and $\|I(a_w, Q)\|$ as in the proof of Lemma 6.4, we have

$$\begin{aligned}
(2) &\leqslant \sum_{\substack{Q \in \mathscr{Q} \\ Q \ni x}} (\exp \|Q\|) \sum_{\substack{a_w \in \mathscr{E}_L \\ \pi(a_w) \subset Q}} \exp[-\zeta \|I(a_w, Q)\|] \\
&\quad \times \prod_{\alpha} \prod_{\substack{j \in I(a_w, Q) \\ j \in \mathbb{Z}^d_{1/2}}} \sum_{k=0}^{\infty} \Bigg(\sum_{\substack{C \in \mathscr{C}_\alpha \\ C \ni j}} \{\exp[(\omega + \zeta) \|C\|]\}\, |f^{\alpha}_{L, I(a_w)}(C)| \Bigg)^k \\
&\leqslant \sum_{\substack{Q \in \mathscr{Q} \\ Q \ni x}} (\exp \|Q\|) \sum_{\substack{a_w \in \mathscr{E}_L \\ \pi(a_w) \subset Q}} \exp[-(\zeta - c_c \log 4) \|I(a_w, Q)\|] = (3).
\end{aligned} \tag{A.21}$$

The number of interfaces with $\|I \cap \pi^{-1}(Q)\| = n$, such that $\pi(\mathbb{W}(I)) \subset Q$, can be bounded by c^n and the minimal length of such interface is $\|Q\|$. Hence,

$$(3) \leqslant \sum_{\substack{Q \in \mathcal{Q} \\ Q \ni x}} (\exp \|Q\|) \sum_{n=\|Q\|}^{\infty} \exp[-(\zeta - c_c \log 4)\, n]\, c^n$$

$$= \sum_{\substack{Q \in \mathcal{Q} \\ Q \ni x}} (\exp \|Q\|) \frac{\exp[(c_c \log 4 - \zeta + \log c)\, \|Q\|]}{1 - \exp[c_c \log(4) - \zeta + \log c]}$$

$$\leqslant 2 \sum_{\substack{Q \in \mathcal{Q} \\ Q \ni x}} \exp[(1 + c_c \log 4 - \zeta + \log c)\, \|Q\|] = (4). \tag{A.22}$$

To bound the last expression, we will use the estimate

$$|\{Q \in \mathcal{Q} : Q \ni x, \|Q\| = n\}| \leqslant c_{d-1}^n. \tag{A.23}$$

And thus

$$(4) \leqslant 2 \sum_{n=1}^{\infty} c_{d-1}^n \exp[(1 + c_c \log 4 - \zeta + \log c)\, n] \leqslant 1, \tag{A.24}$$

which is easy to see from (A.17) and (A.18). ∎

Proving the previous lemma, we have verified the assumptions of Theorem B.2 for the contour model with contour functional Ψ^L. The bounds we found do not depend on the size of box. Thus, we can work with this contour model also in the case $L = \infty$.

A.3. Proof of Lemma 7.1

Proof of (b). We want to prove that the probability measures $P_L^{\mathcal{I}}$ converge weakly to the probability measure $P^{\mathcal{I}}$ on $\mathcal{I}$. According to Lemma 6.1, the probability measures $P_L^{\mathcal{I}}$ can be thought of as the measures on the set $\mathscr{E}_L$. Let us observe that the set of all subsets of $\mathscr{W}$ may be identified with the compact metric space $\{0, 1\}^{\mathscr{W}}$. Endowing it with its Borel σ-algebra, the sets $\mathscr{E}$ and $\mathscr{E}^a$ may be considered as measurable subspaces of the space of subsets of the set of all standard walls. That is why $P^{\mathcal{I}}$ is uniquely determined by its values on the sets of the form

$$\mathscr{M}_{M, \mathbb{W}} = \{\bar{\mathbb{W}} \in \mathscr{E} : \bar{\mathbb{W}} \cap M = \mathbb{W}\} \tag{A.25}$$

for all finite sets $\mathbb{W} \in \mathscr{E}$ and $M \subset \mathscr{W}$. Since

$$\mathscr{M}_{M,\mathbb{W}} = \mathscr{M}_{\mathbb{W}} \Big\backslash \bigcup_{w \in M \setminus \mathbb{W}} \mathscr{M}_{\mathbb{W} \cup \{w\}}, \tag{A.26}$$

one has for any probability P on $\mathscr{E}$ the equality

$$P(\mathscr{M}_{M,\mathbb{W}}) = P(\mathscr{M}_{\mathbb{W}}) - P\Bigg(\bigcup_{w \in M \setminus \mathbb{W}} \mathscr{M}_{\mathbb{W} \cup \{w\}}\Bigg) \tag{A.27}$$

$$= P(\mathscr{M}_{\mathbb{W}}) - \sum_{\substack{\mathbb{W}' \subset M \setminus \mathbb{W} \\ \mathbb{W}' \neq \varnothing}} (-1)^{|\mathbb{W}'|+1} P(\mathscr{M}_{\mathbb{W} \cup \mathbb{W}'}) \tag{A.28}$$

$$= \sum_{\mathbb{W}' \subset M \setminus \mathbb{W}} (-1)^{|\mathbb{W}'|} P(\mathscr{M}_{\mathbb{W} \cup \mathbb{W}'}). \tag{A.29}$$

Thus, to verify the convergence of $P_L^{\mathscr{I}}$ we must only verify the convergence of $P_L^{\mathscr{I}}(\mathscr{M}_{\mathbb{W}})$. We first define the set $\mathscr{F}_L(\mathbb{W})$ as

$$\mathscr{F}_L(\mathbb{W}) = \{S \in \mathscr{F}_L : \mathbb{W}(S) \supset \mathbb{W},\, A \in S \Rightarrow \mathbb{W}(A) \cup \mathbb{W} \neq \varnothing\}. \tag{A.30}$$

One can observe that

$$P_L^{\mathscr{I}}(\mathscr{M}_{\mathbb{W}}) = \sum_{S \in \mathscr{F}_L(\mathbb{W})} \prod_{A \in S} \Psi_L(A) \frac{\mathscr{Z}(\mathbb{A}_L \setminus [[S]], \Psi_L)}{\mathscr{Z}(\mathbb{A}_L, \Psi_L)}, \tag{A.31}$$

where we use $[[S]]$ to denote the set of aggregates not compatible with S. Further, we use the cluster expansions for the nominator and denominator in the last formula. After the standard computation we get

$$P_L^{\mathscr{I}}(\mathscr{M}_{\mathbb{W}}) = \sum_{S \in \mathscr{F}_L(\mathbb{W})} \exp\Bigg(-\sum_{\substack{C \in \mathscr{A}_L^{cl} \\ C \iota S}} \Psi_L^T(C)\Bigg) \prod_{A \in S} \Psi_L(A), \tag{A.32}$$

where we use $C \iota S$ to denote the fact that $\pi(C) \cap \pi(S) \neq \varnothing$, i.e., C is incompatible with S. Let $\varepsilon > 0$. We will show that for L large enough the difference between the last expression and

$$\sum_{S \in \mathscr{F}(\mathbb{W})} \exp\Bigg(-\sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C)\Bigg) \prod_{A \in S} \Psi(A) \tag{A.33}$$

is smaller then ε. First, we prove that the contribution of terms with large S is negligible for both sums. In the following computation we use Lemma A.3 and bound (B.4) applied to aggregate model.

$$\sum_{\substack{S \in \mathscr{F}_{\cdot}(\mathbb{W}) \\ \|S\| \geqslant K,\, \|\pi(S)\| \geqslant K'}} \exp\Bigg(-\sum_{\substack{C \in \mathscr{A}_{\cdot}^{cl} \\ C \iota S}} \Psi_{\cdot}^{T}(C)\Bigg) \prod_{A \in S} \Psi_{\cdot}(A)$$

$$\leqslant \sum e^{\|S\|} \prod_{A \in S} \Psi_{\cdot}(A) = \sum \prod_{A \in S} e^{\|A\|} \Psi_{\cdot}(A)$$

$$\leqslant e^{(1-\omega)K - K'} \prod_{x \in \mathbb{W}} \Bigg(1 + \sum_{A \ni x} \Psi_{\cdot}(A)\, e^{\omega \|A\| + \|\pi(A)\|}\Bigg)$$

$$\leqslant e^{(1-\omega)K - K'} 2^{c_c \|\mathbb{W}\|} \leqslant \varepsilon/4, \tag{A.34}$$

if we chose the constants K or K' large, and if q is large enough to allow us to take $\omega > 1$.

Let us denote by $d(\mathbb{W})$ the minimal L such that $\mathbb{W} \subset U_1(\Lambda_L)$. From now, we consider only L such that $L \geqslant d(\mathbb{W}) + \max(K, K')$. One can observe that

$$\sum_{\substack{S \in \mathscr{F}_L(\mathbb{W}) \\ \|S\| \leqslant K,\, \|\pi(S)\| \leqslant K'}} f(S) = \sum_{\substack{S \in \mathscr{F}(\mathbb{W}) \\ \|S\| \leqslant K,\, \|\pi(S)\| \leqslant K'}} f(S) \tag{A.35}$$

for arbitrary function f. Denoting the sum in last expression by $\sum_s$, we have to prove that

$$\Bigg|\sum_s \exp\Bigg(-\sum_{\substack{C \in \mathscr{A}_L^{cl} \\ C \iota S}} \Psi_L^T(C)\Bigg) \prod_{A \in S} \Psi_L(A)$$

$$-\sum_s \exp\Bigg(-\sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C)\Bigg) \prod_{A \in S} \Psi(A)\Bigg| \leqslant \varepsilon/2. \tag{A.36}$$

We start by showing that skipping the dependence on L in the exponential of first term produces only negligible error if L is large.

$$\Bigg|\exp\Bigg(-\sum_{\substack{C \in \mathscr{A}_L^{cl} \\ C \iota S}} \Psi_L^T(C)\Bigg) - \exp\Bigg(-\sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C)\Bigg)\Bigg|$$

$$\leqslant e^{\|S\|} \Bigg| \sum_{\substack{C \in \mathscr{A}_L^{cl} \\ C \iota S}} \Psi_L^T(C) - \sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi_L^T(C) + \sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi_L^T(C) - \sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C) \Bigg|, \tag{A.37}$$

where we used the fact that $|e^x - e^y| \leqslant e^{\max(x,y)} |x - y|$.

The first two sums differ only on the clusters not in $\mathscr{A}_L^{cl}$. That means that size of C is at least $L - d(\mathbb{V})$. From this we can see that the difference between first two terms is of order $e^{-\omega L}$. The second two terms are also very close. Actually, $\Psi_L^T(C) = \Psi^T(C)$ for C living in the cylinder of size L. The rest can be bounded in the same way as the difference of first two terms. Since the sum $\sum_s$ has only finite number of terms, we have, for L large enough,

$$\left| \sum_s \exp\left(- \sum_{\substack{C \in \mathscr{A}_L^{cl} \\ C \iota S}} \Psi_L^T(C) \right) \prod_{A \in S} \Psi_L(A) \right.$$
$$\left. - \sum_s \exp\left(- \sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C) \right) \prod_{A \in S} \Psi_L(A) \right| \leqslant \varepsilon/4. \tag{A.38}$$

It remains to prove that

$$\left| \sum_s \prod_{A \in S} \Psi_L(A) \exp\left(- \sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C) \right) - \sum_s \prod_{A \in S} \Psi(A) \exp\left(- \sum_{\substack{C \in \mathscr{A}^{cl} \\ C \iota S}} \Psi^T(C) \right) \right| \tag{A.39}$$

can be made arbitrary small. However, we have

$$\sum_s \exp\left(- \sum_{C \cup S \neq \varnothing} \Psi^T(C) \right) \left(\prod_{A \in S} \Psi_L(A) - \prod_{A \in S} \Psi(A) \right) = 0 \tag{A.40}$$

for L larger than $d(\mathbb{W}) + \max(K, K')$ by definition of $\Psi_L(A)$.

Proof of (a). To prove the claim (a) one uses essentially the same computation as in (A.34). The value of q has to be taken large enough to have $\bar{c} = \omega - 1 - c_c \log 2 > 0$. Note, that $\bar{c}$ can be made arbitrary large, if q increases.

Proof of (c). It is simple to notice that $P^{\mathscr{I}}(\mathscr{E}_L) = 1$, since the set $\mathscr{W} \setminus \mathscr{E}$ is covered by a countable union of sets of collections of walls that are "incompatible at some $x \in \mathbb{Z}_{1/2}^d$."

To verify that $P^{\mathscr{I}}(\mathscr{E}^a) = 1$ we use the standard argument. First, we consider the half-line p from a fixed $x \in I_0$, $x \in \mathbb{Z}_{1/2}^d$ parallel to a fixed coordinate axis in $\mathbb{Z}^d$ such that $p \subset I_0$ and the wall w such that $x \in \operatorname{Int} \pi(w)$ and $\|w\| = n$. There is at most $2n$ points in $p \cap \mathbb{Z}_{1/2}^d$ that can be inside of $\pi(w)$, and thus

$$P^{\mathscr{I}}(\{ \mathbb{W} \mid x \in \operatorname{Int} \pi(w), \|w\| = n, w \in \mathbb{W} \})$$
$$\leqslant 2n P^{\mathscr{I}}(\{ \mathbb{W} \mid x \in \pi(w), \|w\| = n, w \in \mathbb{V} \}). \tag{A.41}$$

Hence, the probability that a site x is inside at least n walls may be bounded by

$$\sum_{m=n}^{\infty} P^{\mathscr{I}}(\{ \mathbb{W} : x \in \operatorname{Int} \pi(w), \|w\| = m, w \in \mathbb{W} \})$$

$$\leqslant \sum_{m=n}^{\infty} 2m \sum_{\substack{w : \pi(w) \ni x \\ \|w\| = m}} P_L^{\mathscr{I}}(\{w\})$$

$$\leqslant \sum_{m=n}^{\infty} 2mc_{d-1}^m \exp[-\bar{c}m] \leqslant \sum_{m=n}^{\infty} 2^m c_{d-1}^m \exp[-\bar{c}m]$$

$$= \frac{\exp[-(\bar{c} - \log 2c_{d-1})\, n]}{1 - \exp[-(\bar{c} - \log 2c_{d-1})]}, \tag{A.42}$$

since the length of the nth wall "encircling" i is at least n. Then the probability that the site i is "encircled" by an infinite number of walls is bounded by

$$\lim_{n \to \infty} \frac{\exp[-(\bar{c} - \log 2c_{d-1})\, n]}{1 - \exp[-(\bar{c} - \log 2c_{d-1})]} = 0, \tag{A.43}$$

if $\bar{c} > \log 2c_{d-1}$. From this we see that $P^{\mathscr{I}}(\mathscr{E}^a) = 1$. ∎

APPENDIX B: POLYMER MODELS

We will summarize here the standard facts about contour (polymer) models and cluster expansion. Throughout the whole Appendix A we follow the article.[14]

We will consider a countable set $\mathscr{K}$ of polymers. Let $\imath$ be a reflexive and symmetric relation. We call a pair γ_1, γ_2 incompatible (compatible) if and only if $(\gamma_1, \gamma_2) \in \imath$ $((\gamma_1, \gamma_2) \notin \imath)$. We use the notation $\gamma_1 \imath \gamma_2$ for incompatible polymers. By $\mathscr{D}$, $\mathscr{D}^f$ we denote the set of all (finite) collections $\partial \subset \mathscr{K}$ of mutually compatible polymers. Considering a contour functional Φ: $\mathscr{K} \mapsto \mathbb{C}$, we use $\Phi(\partial) = \prod_{\gamma \in \partial} \Phi(\gamma)$ for each $\partial \in \mathscr{D}^f$

For any finite $\mathscr{L} \subset \mathscr{K}$ we introduce the partition function

$$\mathscr{Z}(\mathscr{L}, \Phi) = \sum_{\partial \in \mathscr{D}(\mathscr{L})} \Phi(\partial), \tag{B.1}$$

where $\mathscr{D}(\mathscr{L}) = \{\partial \in \mathscr{D} \mid \partial \subset \mathscr{L}\}$.

For any $C \subset \mathscr{K}$ we write $\gamma \iota C$ if there is $\bar{\gamma} \in C$ such that $\gamma \iota \bar{\gamma}$. We call C cluster if it is not decomposable into two nonempty sets, $C = C_1 \cup C_2$, such that every pair $\gamma_1 \in C_1$, $\gamma_2 \in C_2$ is compatible. The set of all clusters will be denoted by $\mathscr{C}$.

Theorem B.1. Let functions $a\colon \mathscr{K} \mapsto [0, \infty)$, $l\colon \mathscr{K} \mapsto [0, \infty)$, $\Phi\colon \mathscr{K} \mapsto \mathbb{C}$, and a number $\omega \geqslant 0$ be such that

$$\sum_{\bar{\gamma}\,:\,\bar{\gamma}\iota\gamma} e^{a(\bar{\gamma})+\omega l(\bar{\gamma})}\,|\Phi(\bar{\gamma})| \leqslant a(\gamma) \tag{B.2}$$

for each $\gamma \in \mathscr{K}$. Then $\mathscr{Z}(\mathscr{L}, \Phi) \neq 0$ for every finite $\mathscr{L} \subset \mathscr{K}$ and there exists a unique function $\Phi^T\colon \mathscr{D} \mapsto \mathbb{C}$ such that

$$\log Z(\mathscr{L}, \Phi) = \sum_{C \subset \mathscr{L}} \Phi^T(C) \tag{B.3}$$

for each finite $\mathscr{L} \subset \mathscr{K}$. Moreover, the estimate

$$\sum_{C\iota\gamma} |\Phi^T(C)|\, e^{\omega l(C)} \leqslant a(\gamma) \tag{B.4}$$

holds for each $\gamma \in \mathscr{K}$ with $l(C) \leqslant \sum_{\gamma \in C} l(\gamma)$ and we have $\Phi^T(C) = 0$ whenever $C \notin \mathscr{C}$.

For proof see ref. 8 or ref. 14 and their references.

The last theorem is used to control the behaviour of the aggregate contour model. To simplify our work with contour models that are obtained in Lemma 3.2, we state another theorem, that can be partly proven using the previous one. Actually, it is a simple modification of Theorem B.2 from ref. 8 different numerical factors in formulas come from a more complicated geometry of contours, more precisely from the fact, that contours do not live on a hyper-cubic lattice.

Theorem B.2. Let $|\Phi_\alpha(\gamma)| \leqslant e^{-\tau \|\gamma\|}$ for each $\gamma \in \mathscr{K}$ with $\tau \geqslant 1 + \log(3c)$. Then the statement of Theorem B.1 is fulfilled with the estimate (B.4) replaced by

$$\sum_{\substack{C \in \mathscr{C}_q \\ C \ni i}} |\Phi^T_\alpha(C)|\, e^{\omega \|C\|} \leqslant 1 \tag{B.5}$$

whenever $\omega \leqslant \tau - [1 + \log(3c) + m_\alpha^{-1} \log m_\alpha]$.

Assuming further that Φ is translation invariant, one has

$$\log \mathscr{Z}(\mathscr{K}_\alpha(\Lambda), \Phi_\alpha) = p(\Phi_\alpha)\, |\mathbb{B}(\Lambda)| - \sum_{\substack{C \in \mathscr{C}_\alpha \\ \mathbb{C} \cap (\mathbb{B}(\Lambda))^c \neq \emptyset}} \Phi_\alpha^T(C) \frac{|C \cap \mathbb{B}(\Lambda)|}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \tag{B.6}$$

for each finite $\Lambda \subset \mathbb{Z}^d$, with

$$p(\Phi_\alpha) = 2 \sum_{\substack{C \in \mathscr{C}_\alpha \\ C \ni i}} \frac{\Phi_\alpha^T(C)}{|C \cap \mathbb{B}(\mathbb{Z}^d)|} \tag{B.7}$$

and

$$|\log \mathscr{Z}(\mathscr{K}_\alpha(\Lambda), \Phi_\alpha) - p(\Phi_\alpha)|\, |\mathbb{B}_\Lambda| \leqslant [\exp(-\omega m_\alpha)]\, |\partial \mathbb{B}_\Lambda|. \tag{B.8}$$

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REFERENCES

1. R. L. Dobrushin, Gibbs state describing coexistence of phases for a three-dimensional Ising model, *Teor. Ver. Pril.* **17**:619–639 (1972); English transl. in *Theory Probab. Appl.* **17**:582–600 (1972).
2. L. Laanait, A. Messager, S. Miracle-Sole, J. Ruiz, and S. Shlosman, Interfaces in the Potts model I: Pirogov–Sinai theory of the Fortuin–Kasteleyn representation, *Comm. Math. Phys.* **140**:81–91 (1991).
3. A. Messager, S. Miracle-Sole, J. Ruiz, and S. Shlosman, Interfaces in the Potts model II: Antonov's rule and rigidity of the order disorder interface, *Comm. Math. Phys.* **140**:275–290 (1991).
4. C. Borgs and J. T. Chayes, The covariance matrix of the Potts model: A random cluster analysis, *J. Statist. Phys.* **82**:1235–1297 (1996).
5. G. R. Grimmett, Percolative problems, in *Probability and Phase Transition*, G. R. Grimmett, ed. (Kluwer, Dordrecht), pp. 69–86.
6. G. R. Grimmett, The stochastic random cluster process and the uniqueness of random cluster measures, *Ann. Probab.* **23**:1461–1510 (1995).
7. M. Biskup, C. Borgs, J. T. Chayes, and R. Kotecký, Gibbs states of graphical representations of the Potts model with external fields, *J. Math. Phys.* **41**:1170–1210 (2000).
8. P. Holický, R. Kotecký, and M. Zahradník, Rigid interfaces for lattice models at low temperatures, *J. Statist. Phys.* **50**:755–812 (1988).
9. S. Pirogov and Ya. Sinai, Phase diagrams of classical lattice systems, *Theoret. and Math. Phys.* **25**:1185–1192 (1975) and **26**:39–49 (1976).

10. C. Borgs, R. Kotecký, and S. Miracle-Solé, Finite-size scaling for Potts models *J. Statist. Phys.* **62**:529–552 (1991).
11. H.-O. Georgii, O. Häggström, and C. Maes, *The Random Geometry of Equilibrium Phases*, Phase Transitions, Vol. 18 (Academic Press, 2001).
12. P. Holický, R. Kotecký, and M. Zahradník, Phase diagram of horizontally invariant Gibbs states for lattice models, *Ann. Henri Poincaré* **3**:203–267 (2002).
13. C. E. Pfister and K. Vande Velde, Almost sure quasilocality in the random cluster model, *J. Statist. Phys.* **79**:765–774 (1995).
14. R. Kotecký and D. Preiss, Cluster expansions for abstract polymer models, *Comm. Math. Phys.* **103**:491–498 (1986).